School of Finance

University of St.Gallen

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MATTHIAS D. AEPLI
KARL FRAUENDORFER
ROLAND FÜSS
FLORENTINA PARASCHIV

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Multivariate Dynamic Copula Models: Parameter Estimation and Forecast Evaluation

Matthias D. Aepli* Karl Frauendorfer† Roland Füss‡
Florentina Paraschiv§

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Abstract

This paper introduces multivariate dynamic copula models to account for the time-varying dependence structure in asset portfolios. We firstly enhance the flexibility of this structure by modeling regimes with multivariate mixture copulas. In our second approach, we derive dynamic elliptical copulas by applying the dynamic conditional correlation model (DCC) to multivariate elliptical copulas. The best-ranked copulas according to both in-sample fit and out-of-sample forecast performance indicate the importance of accounting for time-variation. The superiority of multivariate dynamic Clayton and Student-t models further highlight that tail dependence as well as the capability of capturing asymmetries in the dependence structure are crucial features of a well-fitting model for an equity portfolio.

JEL Classification: $C32$, $C51$, $C53$
Keywords: *Multivariate dynamic copulas; regime-switching copulas; dynamic conditional correlation (DCC) model; forecast performance; tail dependence.*

*University of St.Gallen, Institute for Operations Research and Computational Finance, Bodanstrasse. 6, CH-9000 St. Gallen, Switzerland, matthias.aepli@gmail.com.

†University of St.Gallen, Institute for Operations Research and Computational Finance, Bodanstrasse 6, CH-9000 St. Gallen, Switzerland, Phone: +41 (0)71 224 21 05, Email: karl.frauendorfer@unisg.ch.

‡*Corresponding author*: University of St. Gallen, Swiss Institute of Banking and Finance, Rosenbergstrasse 52, CH-9000 St. Gallen, Switzerland and Research Associate at the Centre for European Economic Research (ZEW), Phone: +41 (0)71 224 7055, Fax: +41 (0)71 224 7088, Email: roland.fuess@unisg.ch.

§University of St.Gallen, Institute for Operations Research and Computational Finance, Bodanstrasse 6, CH-9000 St. Gallen, Switzerland, Phone: +41 (0)71 224 30 81, Email: florentina.paraschiv@unisg.ch.

1 Introduction

Financial risks emerge from the volatilities and the dependence structure of the assets comprised in a portfolio. Both elements are often estimated simultaneously in a historical covariance matrix, which in standard statistical approaches is mostly assumed to be constant over time. Traditionally, the most widely used measure in theoretical finance is Pearson's correlation coefficient, which is a natural measure of dependence for elliptical distributions but comes with several limitations. For instance, the correlation is dependent on the marginal distributions and is not invariant under non-linear strictly increasing transformations of the marginals. Furthermore, correlation is unable to capture the entire dependence structure, misses non-linear relations, and does not allow for dependency asymmetries. However, there is strong evidence that the univariate distributions of many financial time series are non-normal and significantly fat-tailed (see, e.g., McNeil, Frey, and Embrechts (2005), Christoffersen (2012)). Assuming a multivariate normal distribution in a non-elliptical world thus neglects the joint extreme events due to the symmetry and its incapability to capture tail dependence (Braun (2011); Embrechts, McNeil, and Straumann (2002)).

With regards to volatility, it is recognized in the literature that financial return series are often heteroscedastic showing alternating clusters of high and low volatility over time. Many scholars further provide evidence of volatility asymmetries, which means that negative news have a larger impact on volatility than positive news (see, e.g., Schwert (1989); Brandt and Kang (2004); Liu (2007)). Thus, the literature encourages the use of an asymmetric GARCH approach to model the time-varying conditional variances of univariate distributions and asymmetries in volatility.

With regards to dependence, there is substantial evidence suggesting asymmetries in the dependence structure, as negative returns are found to be more dependent than positive returns (see, e.g., Erb, Harvey, and Viskanta (1994), Longin and Solnik (2001), Ang and Bekaert (2002), and Ang and Chen (2002)). A large body of literature further suggests that the dependence structure of financial variables is not constant over time, but changes in shape and intensity (see, e.g., Patton (2006b); Longin and Solnik (2001); Ang and Bekaert (2002); Christoffersen (2009)). These features are of prime concern for risk management, since inappropriate models for the relationships between financial variables have been identified as an important element of the recent financial crisis (Stöber and Czado (2012); Financial Services Authority (2009)). The Basel Committee on Banking Supervision (2011, p.10) concludes that for calculating value-at-risk (VaR) measures "time-varying correlations should be taken into account." Yet, the best way to model asymmetries and time-variations of the dependence structure is still an open question (Dias and Embrechts (2010)).

We use copulas to account for these characteristics of financial time series within an asset portfolio framework. The copula theory allows to separately specify the dependence structure as well as the marginals, and provides entire freedom in combining different marginal distributions (Sklar (1959)). These features overcome the limitations of Pearson's correlation coefficient. Most of the studies applying copulas to financial time series data, however, assume the dependence structure to be constant over time (Manner and Reznikova (2012)). This seems unrealistic in the light of the empirical evidence of time-varying dependencies. Patton (2006b) was among the first to address this issue by allowing copulas to be time-varying and therewith triggered a new and fast-growing line of research. Some of the recent contributions which employ copulas to capture time-varying dependencies are Okimoto (2008), Ng (2008), Guégan and Zhang (2010), Dias and Embrechts (2010), Silva Filho, Ziegelmann, and Dueker (2012) and De Lira Salvatierra and Patton (2013). However, these studies and the majority of research on copulas are conducted on the *bivariate* level. This occurs as numerical issues with bivariate copulas are mostly uncritical and their density function can still be graphically depicted without the loss of information. Yet, most portfolios of financial assets contain more than two constituents. With higher dimensions, it becomes increasingly complicated to find a numerically tractable model that is flexible enough to capture real data behavior (Embrechts and Hofert (2014)).

To account for higher dimensions, Savu and Trede (2010) and Okhrin, Okhrin, and Schmid (2008) introduce hierarchical Archimedean copulas, while Oh and Patton (2015) propose factor copula models. In addition, vine copula constructions proposed by Aas, Czador, Frigessi, and Bakken (2009) represent a more flexible approach allowing to estimate the large number of parameters sequentially. Similarly, the literature offers different ways to specify time-varying copulas: beside the regime-switching models, the model developed by Hafner and Manner (2012) and Almeida and Czado (2012) is based on copula parameters which are transformed by a latent Gaussian autoregressive process. A combination of the two approaches is used by Chollete, Heinen, and Valdesogo (2009) and Heinen and Valdesogo (2008) to estimate regime-switching or DCC driven vine copulas with the time-varying feature. Almeida, Czado, and Manner (2012) extend the stochastic autoregressive copula (SCAR) model to higher dimensions using D-vine SCAR models. However, the latter models are estimated sequentially based on a simulated maximum likelihood estimator.

This paper contributes to the literature by introducing multivariate copula models which are capable of accurately capturing and forecasting real portfolio dynamics. To broaden the limited choice of copulas for higher dimensions, available multivariate copulas are amalgamated into multivariate mixture copulas. The idea is based on Nelsen (2006) and Hu (2006), who created "new" dependence models with convex combinations of existing bivariate copulas. Uniting the diverse features of the enclosed copulas, these mixture models provide an increased flexibility to adapt to the data.

To account for time-variations in the dependence structure, we follow two different modeling approaches. The first approach is based on Chollete, Heinen, and Valdesogo (2009), who proposed a regime-switching copula to capture the variations of the dependency structure over time. In this setting, copulas are static within one regime, but differ across regimes. Since the switches between the regimes cannot be known in advance, they are assumed to be governed by a latent Markov process. We further enhance the flexibility of this structure by modeling regimes with multivariate mixture copulas. The second approach to model time-varying dependencies consists of dynamic copulas whose parameters are allowed to vary with every discrete time step. To create dynamic elliptical copulas, the dynamic conditional correlation (DCC) model of Engle (2002) is applied to multivariate elliptical copulas. To build dynamic Archimedean copulas, we follow Braun (2011) in extending Patton (2006b)'s dynamic bivariate Archimedean copula model to higher dimensions. Finally, dynamic convex combinations of different dynamic copulas yield multivariate mixture models which are capable to vary the dependence structure in both intensity and shape over time.

To study the importance of asymmetries and time-variation in the dependence structure of an international stock portfolio, we analyze the in-sample fit and predictive power of the above-mentioned copula models. The performance of the various models is ranked and discussed in the context of Basel traffic light system classification. According to the information criteria, the most suitable dynamic dependence model is the dynamic Student-t-Clayton mixture copula, followed by the dynamic Student-t copula and the dynamic Gaussian-Clayton mixture. The best-ranked copulas according to out-of-sample forecast performance confirm that dynamic copulas generally produce superior forecast accuracy compared to both static and regime-switching copula models. The dynamic Clayton model shows the best forecasts for the lower tail of the profit and loss distribution, which highlights the importance of modeling tail dependence and time-varying dependence structure. In comparison to the multivariate normal model, the dynamic Clayton copula also scales down significantly the number of VAR(99%) violations during the 2007/08 financial crisis period, which confirms its superiority among dynamic copula models.

The remainder of the paper is organized as follows: In the next section, we present our regime-switching and dynamic copula methodologies. Section 3 shows the descriptive statistics of an international stock market portfolio and conducts in-sample analyses by ranking the fits of different copula model specifications. Section 4 investigates the forecast accuracy for the stock portfolio's risk by providing out-of-sample backtests. In addition, it evaluates the models' predictive power during the 2007/08 financial crisis as well as the European sovereign debt crisis. Section 5 concludes.

2 Multivariate Dynamic Copula Models

Copula theory is based on the contribution of Sklar (1959), who showed that a multivariate distribution can be divided into its d marginal distributions and a d-dimensional copula, which completely characterizes the dependence structure between the variables. His theorem provides an accessible way to build valid multivariate distributions from known marginals.

Consider $F(y_1, \ldots, y_d)$ to be a continuous d-variate cumulative distribution function with univariate margins $F_i(y_i)$. Sklar's theorem states that there exists a function C named a copula, which maps $[0,1]^d$ into $[0,1]$ such that

$$F(y_1, \ldots, y_d) = C(F_1(y_1), \ldots, F_d(y_d)). \tag{1}$$

Forecasting in a multivariate setting is based on an extension of Sklar's theorem (1) for conditional joint distributions presented in Patton (2006b). Considering some information set $\mathcal{F}_{t-1}$, Patton shows that the conditional distribution $F(y_1, \ldots, y_d|\mathcal{F}_{t-1})$ can be decomposed into its conditional marginal distributions and the conditional copula such that

$$F(y_1, \ldots, y_d|\mathcal{F}_{t-1}) = C(F_1(y_1|\mathcal{F}_{t-1}), \ldots, F_d(y_d|\mathcal{F}_{t-1})|\mathcal{F}_{t-1}). \tag{2}$$

The d-dimensional conditional copula is:

$$C\left(u_{1,t},...u_{d,t}|\mathcal{F}_{t-1}\right) = F(F_1^{-1}\left(u_{1,t}|\mathcal{F}_{t-1}\right), \dots, F_d^{-1}\left(u_{d,t}|\mathcal{F}_{t-1}\right)). \tag{3}$$

A valid conditional multivariate distribution based on Sklar's theorem and Patton's extension can thus be created by first estimating the models for each of the conditional marginal distributions, $F_i(y_i|\mathcal{F}_{t-1})$, $i = 1,..,d$, construct the probability integral transformed variables $u_{i,t} = F_i(y_{i,t}|\mathcal{F}_{t-1})$, $i = 1,..,d$, and then consider copula models for the joint distribution of these variables. In analogy to the construction of unconditional copulas, this procedure yields a valid d-dimensional model without the intricacy of a simultaneous specification and estimation.[1]

2.1 Regime-Switching Copulas

There is a common sense in the literature that in times of crises the dependence structure between assets increases compared to "normal" times. One approach to account for the different levels of dependence is to switch between different copula models. For instance, Stöber and Czado (2012) show that there are structural breaks in the dependence structure of financial variables similar to the clusters in univariate volatilities. Combinations of regime-switching models with bivariate copulas were proposed for example by Okimoto (2008), Rodriguez (2007) and Silva Filho, Ziegelmann, and Dueker (2012), who estimate regime-switching copulas for bivariate international stock market data.

However, in our approach dependence structures exceed the bivariate level. We choose the methodology of Chollete, Heinen, and Valdesogo (2009), Garcia and Tsafack (2011) and Braun (2011) which employ two dependence regimes that are different in intensity and/or shape. The marginal distributions are modeled separately from the

[1] We do not discuss the main copula models such as the elliptical copulas (Gaussian and Student-t copulas), the Archimedean copulas (Clayton and Frank copulas), and mixture copulas. For an overview of these static copula models we refer the reader to the Internet Appendix A.

dependence structure and are thus not dependent on the regime. This approach allows applying separate copulas for different dependence regimes. Accordingly, the parameters and the families of the copulas remain constant within a regime but differ across the regimes. Switching between the regimes is governed by a latent Markov process which determines the regime probabilities.

To model the dynamics of the data, we follow Hamilton (1989), who proposes a method which allows switching between different density functions. While Hamilton considered univariate time series, our approach focuses on the joint density of multiple time series as described by the copula functions. Since the modeled copulas only diverge with regards to their dependence characteristics, the impact of the different regimes is concentrated on the dependence structure. The model thus expresses different fractions of the joint data density by separate copula functions.

Conditional on being in regime j, the data density is

$$f\left(Y_t|Y_{t-1,s_t=j}\right) = c^j\left(F_1 y_{1,t}, \ldots, F_d y_{d,t}; \theta_c^{(j)}\right) \prod_{i=1}^{d} f_i\left(y_{i,t}; \theta_{m,i}\right), \tag{4}$$

where $Y_t = (y_{1,t}, ..., y_{d,t})$, s_t represents the state variable for the regime, $c^{(j)}(.)$ is the copula density function in the regime j with the according parameter set $\theta_{m,i}$, F_i is the distribution and f_i the according density function of the marginal y_t with the parameters $\theta_{m,i}$. Note that j is an index of the copula, but not of the marginal densities. The model assumes the unobserved state variable to be governed by the transition probability matrix

$$P = Pr(s_t = i|s_{t-1} = j) = p_{i|j}, \tag{5}$$

where $p_{i|j}$ derives the probability that state j will be followed by state i. As the Markov chain is latent and thus not observable, we apply Hamilton's (1989) filter. Accordingly, the transition probability matrix drives the regime probabilities which in turn define the density function of the complete dataset. Explicitly, the filtered process for k regimes obeys

$$\xi_{t|t} = \frac{\xi_{t|t-1} \bigodot \delta_t}{1'(\xi_{t|t-1} \bigodot \delta_t)'}, \tag{6}$$

$$\xi_{t+1|t} = P'\xi_{t|t}, \tag{7}$$

$$\delta_t = \begin{pmatrix} c^{(1)}(F_1\left(y_{1,t}|y_1^{t-1}\right), \ldots, F_d\left(y_{d,t}|y_d^{t-1}\right); \theta_c^{(1)} \\ \vdots \\ c^{(k)}(F_1\left(y_{1,t}|y_1^{t-1}\right), \ldots, F_d\left(y_{d,t}|y_d^{t-1}\right); \theta_c^{(k)} \end{pmatrix}, \tag{8}$$

where $\xi_{t|t}$ is a (k x 1) vector with all the regime probabilities at time t, conditional on the observations until time t; 1 is a (k x 1) vector of ones and $\bigodot$ stands for the Hadamard product. The regime probabilities $\xi_{t+1|t}$ at time $t+1$ conditional on all information until time t are captured by the transition probability matrix P. The copula densities at time t, conditional on being in each one of the two regimes are contained in the vector δ_t. While Equation (6) represents a Bayesian updating of the probability to be in a specific regime given all observations δ_t up to the current time, Equation (7) comprises one forward iteration of the Markov chain. With this recursive procedure it is straightforward to forecast the regime probabilities ($\xi_{t+1|t}$).

The filtered system needs initial values for the regime probabilities $\xi_{1|0}$ from which the optimization procedure is started.[2] Iterations over the two Equations (6) and (7) yield the likelihood value

$$\log \mathcal{L}(\theta) = \sum_{t=1}^{T} \log(1'(\xi_{t|t-1} \bigodot \delta_t)). \tag{9}$$

Naturally one would like to test the null hypothesis that there are k regimes versus the alternative of $k+1$ regimes. Using for example $k = 1$ would answer whether there are any regime switches at all, while using $k = 2$ would determine whether the existence of more than two regimes is supported by the data. However, Hamilton (2008) points out that likelihood ratio tests of these hypotheses do not comply with the usual regularity con-

[2]Gray (1996) points out that for a long enough data set the particular choice of initial values for the regime probabilities becomes irrelevant.

ditions. Given for example that there is truly only one regime, the maximum likelihood estimate for the probability of staying in regime one fails to converge to a well-defined population value. The likelihood ratio test does therewith not have the χ^2-limiting distribution. As a solution, Hamilton (2008) proposes to establish model comparisons based on their ability to forecast.

With the estimated transition probabilities, one can form an inference about the dependence regime at date t based on the realized observations at a later date T. In order to calculate these inferences for the regime probabilities, the Kim filter is used, which represents a combination of the Kalman filter and the Hamilton filter, particularly designed for Markov-switching models (see Hamilton (1988, 1989, 1994)).

Accordingly, the inference of the state variable $\xi_{t|T}$ is performed by considering the entire data obtained until date T ($\xi_{t|T}$). When $T < t$, a forecast about the regimes in the future is made, but when $T > t$, the probabilities for the regime at time t are ex post probabilities. According to Kim (1994), these inferences may be calculated with the following iterative algorithm

$$\xi_{t|T} = \xi_{t|t} \bigodot \left(P' \times [\xi_{t+1|T}(\div)\xi_{t+1|t}]\right), \tag{10}$$

where $\bigodot$ and $(\div)$ stand for the Hadamard multiplication or division, respectively. Initiating with the probability $\xi_{T|T}$, obtained from Equation (6) for $t = T$, the process iterates backwards on Equation (10). This procedure is valid only with first-order Markov chains such as the one at hand. With the estimated transition probability matrix (5), the inference may be computed based on the entire information in the sample.

The limitation on a number of different static dependence structures as modeled by the regime-switching copulas may still be too restrictive. To increase the adaptability of the dependence specification one might think of simply increasing the number of regimes. However, a more flexible approach consists in allowing the dependence structure to be dynamic, i.e. vary with every discrete time step.

2.2 Dynamic Copulas

Engle and Sheppard (2001) and Engle (2002) established the basis for the estimation of time-varying dependence structure by introducing the dynamic conditional correlation model. In the field of copulas, the seminal work of Patton (2006b) was among the first to allow copulas to be time-varying.[3]

Dynamic Gaussian Copula. Based on Engle (2002), the correlation matrix Σ_t of the dynamic Gaussian copula is set to evolve through time as follows:

$$Q_t = (1-\alpha-\beta)\bar{Q} + \alpha z_{t-1}z'_{t-1} + \beta Q_{t-1} \tag{11}$$

$$\Sigma_t = \tilde{Q}_t^{-1} Q_t \tilde{Q}_t^{-1}, \tag{12}$$

where z_t is the vector of transformed standardized residuals $z_{i,t} = skewed - t^{-1}_{\nu,\lambda}(u_{i,t})$, $\bar{Q}$ is the sample correlation of z_t and $\tilde{Q}_t = [\tilde{q}_{ii,t}] = [\sqrt{q_{ii,t}}]$ is the diagonal square matrix with the square root of the ith diagonal element of Q_t on its ith diagonal position. The constraints for the parameters α and β are $\alpha + \beta < 1$, with $\alpha, \beta \in (0,1)$.
The dynamic Gaussian copula is therewith defined as:

$$C^{Ga}_{\Sigma_t}(u_1, ...u_d) = \Phi_{\Sigma_t}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)). \tag{13}$$

Dynamic Student-*t* Copula. The Student-t copula parameters consists of the correlations and the degrees of freedom, ν. The dynamic process which drives the correlations is identical to the one defined for the Gaussian copula in Equations (11) and (12). We further allow the degrees of freedom parameter to vary over time (Jin and Lehnert

[3]Some of the contributions to this field on the bivariate level include Guégan and Zhang (2010), Dias and Embrechts (2010) and De Lira Salvatierra and Patton (2013), and on the multivariate level Jin and Lehnert (2011), Braun (2011) and Christoffersen, Errunza, Jacobs, and Langlois (2012).

(2011)). The Student-t copula is therewith not only provided with the capability to adapt the level of dependence, but also the strength of tail dependence over time. Fantazzini (2008) proposes to model the evolution of the degrees of freedom parameter of a bivariate Student-t copula as

$$\nu_t = \Lambda(\varsigma + \varphi|u_{t-1} - v_{t-1}|), \tag{14}$$

where Λ is a logistic transformation designed to keep the conditional degrees of freedom in $(2, 100)$ at all times.

The aim of our study is not to partition the observations into multiple clusters (see Braun (2011)), but to compute the absolute distance (AD) between all observations in time t. Therefore, the number of clusters is set to $k = 1$, which means that the AD_{ℓ_1}-norm is the sum of absolute differences between the observations u_t and their median $\tilde{u}_t$ at time t:

$$AD_{\ell_1} = \sum_{j=1}^{k} \sum_{i=1}^{d} |u_{i,t} - \tilde{u}_{j,t}| = \sum_{i=1}^{d} |u_{i,t} - \tilde{u}_t|. \tag{15}$$

Replacing the bivariate absolute difference in Equation (14) with the multivariate absolute difference AD_{ℓ_1} in (15) yields the dynamic process of the degrees of freedom of a multivariate Student-t copula

$$\nu_t = \Lambda(\varsigma + \varphi \sum_{i=1}^{d} |u_{i,t-1} - \tilde{u}_{t-1}|). \tag{16}$$

The dynamic multivariate Student-t copula is therewith defined as

$$C^t_{\nu_t,\Sigma_t}(u_1, ..., u_d) = t_{\nu_t,\Sigma_t}(t^{-1}_{\nu_t}(u_1), ..., t^{-1}_{\nu_t}(u_d)). \tag{17}$$

Dynamic Archimedean Copulas. Patton (2006b) adapts the idea of Engle (2002) to model the dynamics of bivariate Archimedean copulas with an ARMA-type process. He assumes that the functional form of the copula stays fixed over the sample, whereas the transformed copula parameter as Kendall's tau varies according to the evolution equation

$$\rho_{\tau_t} = \Lambda\left(\omega + \beta \cdot \rho_{\tau_{t-1}} + \alpha \cdot \frac{1}{10}\sum_{j=1}^{10} |u_{t-j} - v_{t-j}|\right), \tag{18}$$

where $\Lambda(x) = (1 + e^{-x})^{-1}$ is the logistic transformation to keep $\rho_{\tau_t} \in [0, 1]$ at all times and (u_t, v_t) are two observations at time t.[4] The Clayton[5] respectively the Frank copula[6] parameter in time t can then be obtained using the functional relationship between Kendall's tau and the Archimedean copula parameter

$$\begin{array}{ll} \text{Copula} & \rho_\tau \\ C_\theta^{Cl} & \theta/(\theta+2) \\ C_\theta^{Fr} & 1 - 4\theta^{-1}(1 - D_1(\theta)), \end{array} \tag{19}$$

where $D_1(\theta)$ is the Debye function of order one $D_1(\theta) = \theta^{-1}\int_0^\theta t/(\exp(t) - 1)dt$ (Hofert, Mächler, and McNeil (2013)).

The dynamic process of Patton (2006b) in (18) is yet again limited to bivariate applications through the absolute difference term $|u_{t-1} - v_{t-1}|$. To extend (18) to the multidimensional world, this difference term is substituted with the multivariate absolute distance AD_{ℓ_1} in (15). This yields a multivariate extension of Patton (2006b)'s parameter evolution process

$$\rho_{\tau_t} = \Lambda\left(\omega + \beta \cdot \rho_{\tau_{t-1}} + \alpha \cdot \frac{1}{10}\sum_{j=1}^{10}\sum_{i=1}^{d} |u_{i,t-j} - \tilde{u}_{t-j}|\right), \tag{20}$$

[4]Kendall's tau is the rank correlation for two vectors of random variables Y_1 and Y_2, defined as $\rho_\tau = E(\text{sign}((Y_1 - \tilde{Y}_1)(Y_2 - \tilde{Y}_2)))$, where $(\tilde{Y}_1, \tilde{Y}_2)$ is an independent copy of (Y_1, Y_2) (McNeil, Frey, and Embrechts (2005)).

[5]The generator function of the Clayton copula is defined as $\phi_{Cl}(u) = \frac{1}{\theta}\left(u^{-\theta} - 1\right)$, where the permissible parameter range is $\theta \in (0, \infty)$. A d-dimensional Clayton copula is given by $C(u_1, ..., u_d) = \left(\sum_{i=1}^d u_i^{-\theta} - d + 1\right)^{-\frac{1}{\theta}}$ (see Internet Appendix A.2).

[6]The Frank copula generator is given by $\phi_{Fr}(u) = \log\left(\frac{\exp(-\theta u)-1}{\exp(-\theta)-1}\right)$, hence $\phi_{Fr}^{-1}(u) = \frac{1}{\theta}\log\left(1 + e^u(e^{-\theta} - 1)\right)$ is completely monotonic if $\theta \in (0, \infty)$. The multivariate Frank copula is $C(u_1, ..., u_d) = -\frac{1}{\theta}\log\left(1 + \frac{\prod_{i=1}^d (e^{-\theta u_i} - 1)}{(e^{-\theta} - 1)^{d-1}}\right)$ (see Internet Appendix A.2).

where $\tilde{u}_t$ is the median of $u_1, ..., u_d$ in time t and $\Lambda(x) = (1 + e^{-x})^{-1}$. With (20), the parameter of the multivariate Clayton copula in time t, θ_t^{Cl}, is then given in closed form through (19). The Frank copula parameter θ_t^{Fr} in terms of Kendall's tau, however, is not available in closed form but has to be determined numerically. To achieve an efficient calibration of the dynamic Frank copula, we directly model the dynamics of θ_t^{Fr} as

$$\theta_t^{Fr} = \omega + \beta \cdot \theta_{t-1}^{Fr} + \alpha \cdot \frac{1}{10} \sum_{j=1}^{10} \sum_{i=1}^{d} |u_{i,t-j} - \tilde{u}_{t-j}|, \tag{21}$$

where the constraint $\theta_t^{Fr} \geq 0$ ensures that the parameter remains in the permissible range. Stationarity and invertibility is accounted for with the constraints $|\alpha| < 1$ and $|\beta| < 1$.

Dynamic Mixture Copulas. Ng (2008) adopts the dynamic process of Patton (2006a) to create a time-varying specification of the weight in the mixture copula depending on the natural filtration of the process. He suggests a dynamic bivariate mixture copula model, where the parameters of the copulas are constant, but the weighting parameter is stochastic, following an ARMA-type model for the mixture weight:

$$w_{i,t} = \omega_i + \alpha_i \cdot h_{i,t-1}(.) + \beta \cdot w_{i,t-1}. \tag{22}$$

Equation (22) therewith establishes a linear relationship between the mixture weight w_i in time t and the according lagged value in $t-1$ and $h(.)$, which is a stochastic explanatory variable or a special function. In particular, Ng (2008) proposes to model $w_{i,t}$ with the special function being

$$h_{i,t-1}(.) = \frac{1}{10} \sum_{p=1}^{10} |u_{t-p} - v_{t-p}|. \tag{23}$$

However, this model is also limited to the bivariate setting due to the absolute distance measure $|u_{t-p} - v_{t-p}|$. Braun (2011) suggests an extension of Ng (2008)'s concept to higher dimensions by replacing the absolute distance measure with the copula density relative to the sum of all copula densities during the lag period. This results in a special function of the following type:

$$h^c_{i,t-1}(.) = \frac{1}{10}\sum_{p=1}^{10}\left(\frac{c_i(u_{1,t-p},...,u_{d,t-p};\theta_i)}{\sum_{j=1}^{n} c_j(u_{1,t-p},...,u_{d,t-p};\theta_j)}\right). \tag{24}$$

To generate weight forecasts with Equation (24) plugged into (22), Braun (2011) has to impose six different constraints on the weight process parameters in order to keep the resulting weights $w_{i,t}$ within the unit interval. In contrast, we make use of the logistic function $\Lambda(x) = (1+e^{-x})^{-1}$ which in combination with (24) and (22) results in:

$$w_{i,t-1} = \Lambda\left(\omega_i + \alpha_i h^c_{i,t-1}(.) + \beta_i w_{i,t-1}\right). \tag{25}$$

The weight parameters are therewith bounded on the unit interval without the need to impose any constraints on the parameters. Note that Equation (25) also nests the static mixture copula with $\alpha = \beta = 0$. Employing the dynamic weights of Equation (25) in the mixture copula[7] yields the complete multivariate dynamic mixture copula model:

$$C\left(u_1,...,u_d;w_{1,t},...,w_{n,t};\theta_1,...,\theta_n\right) = \sum_{j=1}^{n}[w_{j,t},C_j(u_1,...,u_d,\theta_j)]. \tag{26}$$

It has to be emphasized that the parameters θ_j and $w_{j,t}$ have different functions within the mixture copula construct, allowing a very flexible way of modeling dependence structures. While the association parameter θ controls the degree of dependence, the weight parameter

[7]The applied static mixture copulas consist of two different copulas but of d dimensions and have distribution functions of the form $C\left(u_1,...,u_d;w,\theta\right) = wC_1(u_1,...,u_d,\theta_1) + (1-w)C_2(u_1,...,u_d,\theta_2)$,

where θ_1 and θ_2 are the parameter vectors of the different copulas. The density of the static mixture construct is simply the convex combination of the copula densities involved in the mixture:

$$c\left(u_1,...,u_d;w,\theta\right) = wc_1(u_1,...,u_d,\theta_1) + (1-w)c_2(u_1,...,u_d,\theta_2)$$

(see Internet Appendix A.2).

w determines the structure of the dependence.

The advantage of linking the weight parameter to the copula densities lies in the difference of the copula density functions. The Clayton copula for example is capable of modeling lower tail dependence and exhibits its largest density in the lower tail. Thus, the weight parameter in the dynamic mixture structure is directly coupled with the capabilities of the mixture copula constituents to describe the dependence structure during the lag period.

Shifts in the dependence structure are expected to have an immediate effect on the dynamic weights. A rise of one copula's relative density signalizes its enhanced fit to the current dependence pattern. Through the dynamic weight process in (25), this copula's weight in the mixture construct and its impact on the overall mixture density therewith extends. Calibrating this model using maximum likelihood estimation ensures that the parameters of each copula in the mixture are fitted most accurately to those data fractions, where the dependence structure naturally concurs with the copula's characteristics.[8] Every individual copula thus only captures the dependence in a specific part of the data set in a optimal way, but merging the copulas into a mixture structure governed by the dynamic weight process yields an overall accurate and flexible dependence model.

3 In-Sample Analysis

In this section, we conduct a comparative analysis of the performance of static, regime-switching, and dynamic copulas to investigate the joint dynamics of equity indexes within a portfolio framework. As preliminary analysis, we firstly calibrated three static mixture copulas to the data by combining the asymmetric Clayton copula with three symmetric

[8]We estimate the model parameters and the corresponding standard errors for the regime-switching model and for the dynamic multivariate copulas by using a multi-stage maximum likelihood estimation procedure as shown in Internet Appendix B.

copulas: Gaussian, Student-t, and Frank copulas.[9] In particular, we examined the six months rolling Kendall's tau computed via the one-to-one mapping of Frank's multivariate copula parameter θ_F and Kendall's tau. The analysis of the dependence over time confirms that there are substantial changes in both structure and level of dependence during the observation period. Neglecting this time-variation might result in inaccurate risk forecasts. To capture the dynamics in the dependence structure among a portfolios' assets, the two proposed approaches, regime-switching and dynamic copula model, are evaluated.

3.1 Data and Summary Statistics

We investigate the interactions between international equity indices within a portfolio framework. Based on the central limit theorem, such a portfolio is more likely to exhibit elliptical dependence than a portfolio of individual stocks.[10] All returns are computed with $\log(P_t/P_{t-1})$, where P_t is the value of the index at time t. The data consists of weekly returns from Wednesday to Wednesday, to avoid any day-of-the-week effects. The sample covers the period from June 30, 1988 until June 5, 2013, yielding 1'300 weekly returns. The data covers the following indices: the Swiss SMI, the German DAX 30, the French CAC 40, the British FTSE 100, the US S&P 500, the Canadian TSX 60, the Chinese Hang Seng and the Japanese Nikkei 225.[11] The data has been collected from Bloomberg. We assume a currency hedged USD investor in order to eliminate exchange rate effects and concentrate on the interactions of the assets.

The descriptive statistics of weekly returns of each equity index under consideration are summarized in Internet Appendix B. Each of the equity index return series displays a negative skewness and a leptokurtic distribution. The Jarque and Bera (1987) test rejects the null hypothesis that the sample returns follow a normal distribution for every contin-

[9]Note that a comparative in-sample evaluation of the performance of static mixture copulas is not in the scope of this study, but can be found in the Internet Appendix C.

[10]Note that the prices of some indices in the data set are market capitalization-weighted averages while the classical Central Limit Theorem is based on equally weighted averages.

[11]For the sake of brevity, we employ eight-dimensional models. However, given the scalability of the developed models they are capable of handling portfolios of arbitrary dimensions.

uous return series at the 1%-significance level. The test of Leybourne and McCabe (1999) (LMC) and the Augmented Dickey-Fuller (ADF) test (Dickey and Fuller (1979)) both indicate that each of the return series is stationary.[12] The Ljung an Box (1978) and Engle's Lagrange multiplier statistics show clear evidence for ARCH effects. In order to remove the heteroscedasticity from our return series, each individual risk factor will be modeled by a GARCH specification. The univariate model for each index return series was found by selecting the AIC and BIC optimal model considering ARMA(p,q) specifications for the conditional mean up to order (p=3, q=3) and GARCH(P,Q), EGARCH(P,Q), and GJR-GARCH(P,Q) volatility models up to order (P=3, Q=3). The initial autocorrelation present in the squared returns has been successfully removed by the GARCH models, so that the standardized residuals are now independent and identically distributed. To account for potential asymmetries in the distribution of standardized residuals we employ Hansen's (1994) skewed Student-t distribution.[13]

3.2 Regime-Switching Copulas

This subsection presents the estimation results of the two-regime-switching copula models calibrated to the entire sample data. All copula models are estimated using the same residuals which result from the filtering with the univariate models. The first three regime-switching models combine the elliptical copulas into a two-regime setup. While the Gaussian/Gaussian (G/G) model allows for two regimes with different levels of dependence, it does not capture tail dependence in any of them, whereas the Gaussian/Student-t (G/T) copula allows for tail dependence in one regime. The underlying idea for the latter is that returns may have (presumably tranquil) periods without tail dependence, which may be well described with a Gaussian copula, while the other (presumably turbulent) periods

[12] The first test is defined by not rejecting the null hypothesis of a stationary AR(k) against a non-stationary ARIMA(k,1,1) and the latter by rejecting the null hypothesis of a unit-root against a trend-stationary alternative augmented with k lagged difference terms on the usual significance levels for all orders $k = (1, ..., 20)$.

[13] The results on GARCH specifications and the skewed Student-t distribution of standardized residuals are shown in the Internet Appendix C.1.

with tail dependence are described by the Student-t copula. The Student-t/Student-t (T/T) regime-switching model then allows for tail dependence in both regimes.

As the purely elliptical models are symmetric, the Gaussian/Student-t model is enhanced by mixing one of them with the asymmetric Clayton which firstly results in the Gaussian/Student-t-Clayton mixture (G/TC). This setup is thus capable of capturing different levels of lower and upper tail dependence in one regime. Secondly, the Student-t/Gaussian-Clayton mixture copula (T/GC) allows for symmetric (lower and upper) tail dependence in one regime and an asymmetric dependence with a probability of joint negative extreme returns in the other regime.

Table 1 presents the parameter estimates of two-regime models of which three contain only elliptical copulas. The regimes' parameters are listed in the order indicated by the abbreviated name, i.e. for the G/T regime switch copula, the Gaussian regime parameters are listed under *Regime 1* and the Student-t copula parameters under *Regime 2*. The results indicate that the maximum likelihood estimation procedure identifies a high and a low dependence regime, which is consistent with the findings of Chollete, Heinen, and Valdesogo (2009) and Braun (2011). The copula correlation coefficients in the high-dependent regime are larger for all index pairs. The high-dependence regime features some very large correlations such as for example between the German DAX and the French CAC, where it exceeds 0.9 in every model specification. In contrast, the DAX:CAC copula correlation in the low dependence regime is as low as 0.5. Generally, the highest copula correlations are between the returns of the European equity indices.

The Student-t copula appears to be particularly well capable of describing periods of high dependence, as it is responsible for the high-dependence regime in every structure it is a part of. The fact that the Student-t copula consistently models the high-dependence regimes reflects its versatility due to the parameter plurality (in comparison to the Archimedean copulas) and its capability to model tail dependence (in contrast to the Gaussian copula). Indeed, the degrees of freedom of the Student-t copulas in the mix-

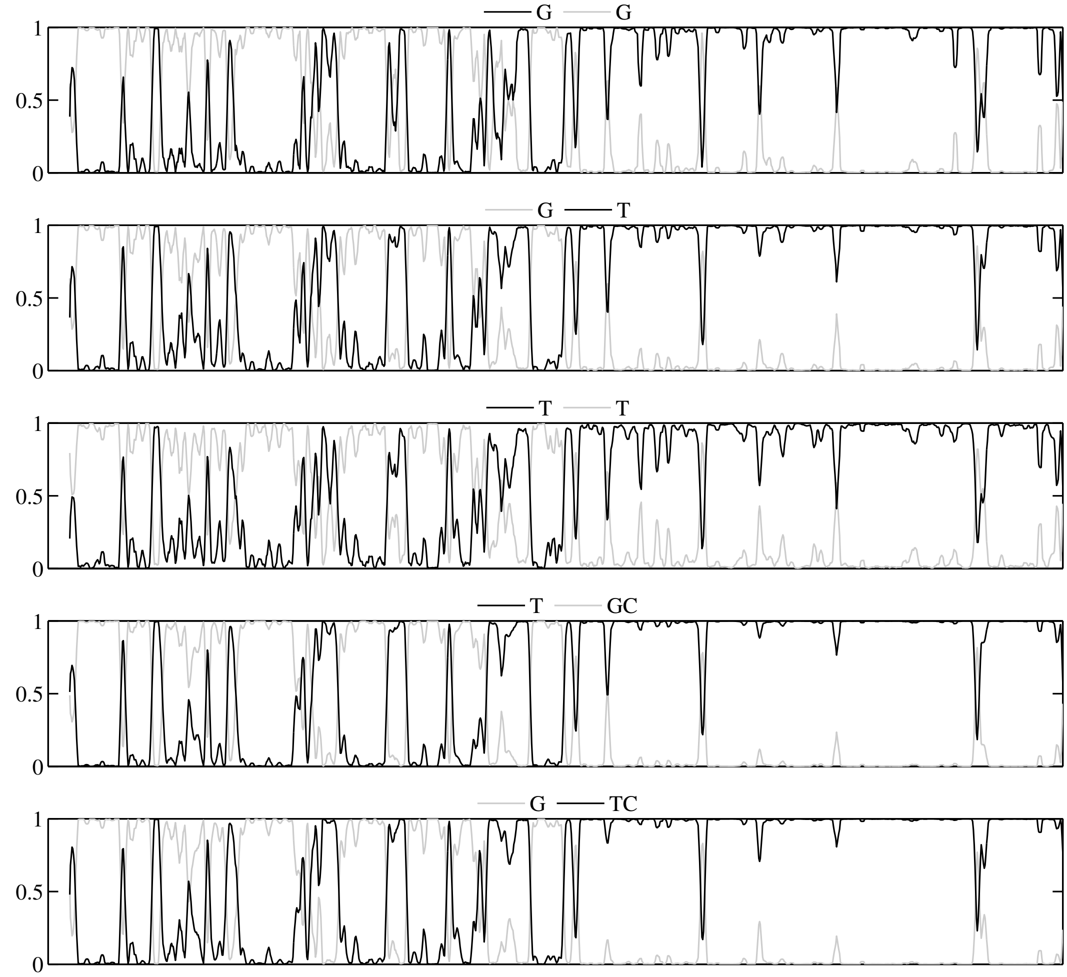
Table 1: Regime-Switching Copula Parameters

Copula	G/G	G/T	T/T	T/GC	G/TC
			Regime 1		
SMI:DAX	0.838 (0.154)	0.496 (0.108)	0.822 (0.125)	0.826 (0.090)	0.452 (0.104)
SMI:CAC	0.858 (0.188)	0.409 (0.141)	0.847 (0.166)	0.853 (0.130)	0.340 (0.139)
SMI:FTS	0.838 (0.179)	0.414 (0.129)	0.820 (0.157)	0.830 (0.139)	0.327 (0.139)
SMI:SPX	0.717 (0.160)	0.408 (0.094)	0.712 (0.102)	0.690 (0.105)	0.381 (0.100)
SMI:TSX	0.614 (0.141)	0.364 (0.081)	0.564 (0.063)	0.583 (0.087)	0.377 (0.084)
SMI:HSE	0.612 (0.220)	0.183 (0.136)	0.561 (0.152)	0.605 (0.160)	0.133 (0.156)
SMI:NIK	0.573 (0.249)	0.086 (0.142)	0.546 (0.170)	0.518 (0.195)	0.093 (0.185)
DAX:CAC	0.932 (0.138)	0.570 (0.100)	0.922 (0.127)	0.919 (0.095)	0.501 (0.100)
DAX:FTS	0.849 (0.185)	0.429 (0.129)	0.821 (0.168)	0.823 (0.128)	0.335 (0.136)
DAX:SPX	0.798 (0.173)	0.406 (0.108)	0.791 (0.140)	0.768 (0.121)	0.386 (0.120)
DAX:TSX	0.657 (0.150)	0.392 (0.081)	0.621 (0.076)	0.644 (0.088)	0.394 (0.089)
DAX:HSE	0.641 (0.193)	0.290 (0.107)	0.602 (0.123)	0.657 (0.135)	0.225 (0.126)
DAX:NIK	0.619 (0.234)	0.170 (0.135)	0.597 (0.154)	0.563 (0.195)	0.193 (0.172)
CAC:FTS	0.883 (0.158)	0.495 (0.115)	0.861 (0.144)	0.873 (0.114)	0.418 (0.116)
CAC:SPX	0.801 (0.185)	0.404 (0.121)	0.804 (0.152)	0.791 (0.138)	0.383 (0.133)
CAC:TSX	0.683 (0.171)	0.380 (0.103)	0.649 (0.097)	0.673 (0.108)	0.338 (0.105)
CAC:HSE	0.646 (0.192)	0.284 (0.105)	0.604 (0.126)	0.650 (0.139)	0.207 (0.132)
CAC:NIK	0.632 (0.238)	0.220 (0.133)	0.613 (0.152)	0.581 (0.188)	0.256 (0.169)
FTS:SPX	0.775 (0.145)	0.517 (0.094)	0.763 (0.113)	0.776 (0.102)	0.460 (0.095)
FTS:TSX	0.708 (0.174)	0.370 (0.103)	0.664 (0.102)	0.687 (0.107)	0.355 (0.104)
FTS:HSE	0.664 (0.191)	0.353 (0.112)	0.623 (0.139)	0.673 (0.138)	0.251 (0.127)
FTS:NIK	0.594 (0.230)	0.209 (0.126)	0.559 (0.147)	0.540 (0.178)	0.244 (0.160)
SPX:TSX	0.751 (0.054)	0.669 (0.032)	0.732 (0.034)	0.744 (0.028)	0.640 (0.026)
SPX:HSE	0.638 (0.186)	0.281 (0.116)	0.593 (0.130)	0.643 (0.132)	0.214 (0.127)
SPX:NIK	0.561 (0.194)	0.249 (0.116)	0.550 (0.131)	0.521 (0.162)	0.296 (0.152)
TSX:HSE	0.548 (0.129)	0.303 (0.082)	0.496 (0.067)	0.554 (0.091)	0.239 (0.090)
TSX:NIK	0.511 (0.166)	0.220 (0.097)	0.472 (0.097)	0.452 (0.123)	0.246 (0.118)
HSE:NIK	0.590 (0.209)	0.186 (0.123)	0.558 (0.135)	0.560 (0.177)	0.196 (0.172)
ν_1			10.218 (0.865)	10.362 (0.989)	
			Regime 2		
SMI:DAX	0.513 (0.154)	0.813 (0.066)	0.511 (0.126)	0.477 (0.134)	0.826 (0.090)
SMI:CAC	0.450 (0.185)	0.846 (0.088)	0.433 (0.168)	0.415 (0.190)	0.853 (0.130)
SMI:FTS	0.425 (0.184)	0.813 (0.085)	0.425 (0.164)	0.368 (0.192)	0.830 (0.139)
SMI:SPX	0.405 (0.158)	0.691 (0.068)	0.424 (0.106)	0.383 (0.132)	0.690 (0.105)
SMI:TSX	0.360 (0.142)	0.563 (0.050)	0.385 (0.068)	0.340 (0.106)	0.583 (0.087)
SMI:HSE	0.190 (0.218)	0.575 (0.092)	0.211 (0.152)	0.110 (0.207)	0.605 (0.160)
SMI:NIK	0.117 (0.250)	0.543 (0.113)	0.144 (0.169)	0.053 (0.222)	0.518 (0.195)
DAX:CAC	0.608 (0.139)	0.917 (0.072)	0.593 (0.131)	0.573 (0.133)	0.919 (0.095)
DAX:FTS	0.438 (0.193)	0.815 (0.092)	0.431 (0.172)	0.422 (0.176)	0.823 (0.128)
DAX:SPX	0.408 (0.169)	0.772 (0.080)	0.413 (0.138)	0.387 (0.156)	0.768 (0.121)
DAX:TSX	0.394 (0.152)	0.611 (0.060)	0.407 (0.077)	0.366 (0.113)	0.644 (0.088)
DAX:HSE	0.291 (0.186)	0.618 (0.085)	0.296 (0.125)	0.231 (0.168)	0.657 (0.135)
DAX:NIK	0.174 (0.233)	0.592 (0.105)	0.203 (0.147)	0.116 (0.214)	0.563 (0.195)
CAC:FTS	0.508 (0.164)	0.861 (0.087)	0.521 (0.149)	0.500 (0.146)	0.873 (0.114)
CAC:SPX	0.411 (0.185)	0.783 (0.087)	0.402 (0.151)	0.386 (0.183)	0.791 (0.138)
CAC:TSX	0.400 (0.176)	0.638 (0.069)	0.392 (0.099)	0.376 (0.142)	0.673 (0.108)
CAC:HSE	0.294 (0.189)	0.625 (0.089)	0.286 (0.131)	0.247 (0.161)	0.650 (0.139)
CAC:NIK	0.210 (0.234)	0.606 (0.107)	0.242 (0.146)	0.140 (0.214)	0.580 (0.188)
FTS:SPX	0.494 (0.149)	0.747 (0.063)	0.485 (0.123)	0.518 (0.126)	0.776 (0.102)
FTS:TSX	0.381 (0.183)	0.663 (0.067)	0.380 (0.111)	0.377 (0.137)	0.687 (0.107)
FTS:HSE	0.368 (0.193)	0.640 (0.084)	0.334 (0.140)	0.341 (0.172)	0.673 (0.138)
FTS:NIK	0.206 (0.226)	0.555 (0.100)	0.223 (0.146)	0.165 (0.194)	0.540 (0.178)
SPX:TSX	0.670 (0.055)	0.742 (0.020)	0.677 (0.037)	0.679 (0.037)	0.744 (0.028)
SPX:HSE	0.283 (0.186)	0.598 (0.075)	0.273 (0.133)	0.249 (0.161)	0.643 (0.132)
SPX:NIK	0.246 (0.191)	0.535 (0.086)	0.243 (0.128)	0.208 (0.169)	0.521 (0.162)
TSX:HSE	0.326 (0.131)	0.513 (0.051)	0.308 (0.065)	0.300 (0.106)	0.554 (0.091)
TSX:NIK	0.224 (0.168)	0.469 (0.071)	0.236 (0.085)	0.200 (0.133)	0.452 (0.123)
HSE:NIK	0.181 (0.199)	0.566 (0.084)	0.181 (0.135)	0.121 (0.187)	0.560 (0.179)
ν_2		9.913 (0.688)	10.434 (0.895)		9.527 (0.681)
θ_C				0.866 (0.400)	2.268 (0.312)
w_C				0.156 (0.068)	0.071 (0.049)
$p_{1\|1}$	0.923 (0.052)	0.868 (0.091)	0.880 (0.052)	0.954 (0.091)	0.913 (0.125)
$p_{2\|2}$	0.891 (0.047)	0.924 (0.050)	0.828 (0.045)	0.924 (0.105)	0.948 (0.071)
$\log \mathcal{L}$	3641	3631	3590	3655	3617
AIC	-7166	-7144	-7060	-7188	-7112
BIC	-6866	-6839	-6750	-6873	-6787

This table shows regime-switching copula parameters with standard errors in parentheses. The copulas are abbreviated as follows: Clayton (C), Gaussian (G), Student-t (T), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). w_C denotes the weight of the Clayton copula in the mixture. The forward slash indicates the separate regimes. $p_{i|i}$ denotes the probability of staying in regime i.

ture structures is between 8.884 and 10.434, which clearly indicates that the returns are tail dependent in the high-dependence regimes. According to both information criteria AIC and BIC, the best fitting model is the T/GC followed by the G/G and the G/T.[14]

Figure 1: Regime Probabilities



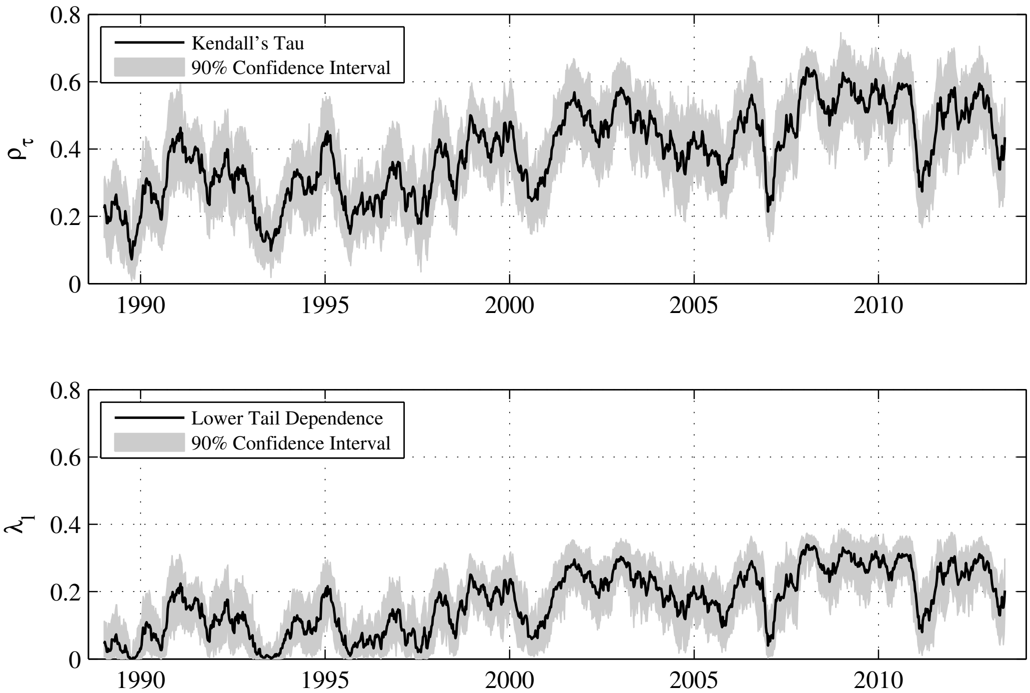
This figure shows Kim filtered regime probabilities of the various regime-switching copula models over the entire sample period. The copulas are abbreviated as follows: Gaussian (G), Student-t (T), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). The solid black line marks the high dependence regime.

Figure 1 depicts the Kim filtered evolution of the state probabilities over the entire

[14] Besides a high and a low dependence state, we also extend the model to three regimes to model some middle or "normal" state of the economy with a dependence level midway between the ones of the extreme states. In contrast to what one might expect, this "normal" state (not shown here) is not modeled by the Gaussian copula but by the Clayton copula in the G/T/C and G/C/F specifications. The low probability of staying in the third regime conditional on being in regime three suggests that this regime is more of a pass-through between the other regimes which both display much higher probabilities $p_{1|1}$, $p_{2|2}$. Furthermore, adding a third regime as done with the G/T/C and G/C/F model does not improve the model fit and the forecast performance substantially (see Internet Appendix C.3).

sample period. All seven models depict a shift from low to high dependence over the observation period. After the year 2001, all models remain in the high-dependence regime (marked with the solid black line). Comparing the Kim filtered regime paths with the rolling Kendall's tau in Figure 2, one can identify the intermediate heights of the average rank correlation around 1991, 1995, and before the turn of the millennium in the regime probabilities of each model, since these heights are captured by a high-dependence regime probability of close to one in every model. Furthermore, for the second half of the period under scrutiny, the high-dependence regime clearly takes over in every regime-switching model, having a regime probability of close to one for the vast majority of the time.

Figure 2: Dependence Level and Lower Tail Dependence over Time



This figure shows Kendall's tau implied by the multivariate Frank copula (upper graph) and lower tail dependence implied by the multivariate Clayton copula (lower graph) of the international equity index portfolio over a six months rolling window along with 90% bootstrap confidence intervals obtained from 500 bootstrap replications of the data.

3.3 Dynamic Copulas

Next, the dynamic copula models as shown in Subsection 2.2 are calibrated to the same residuals obtained by filtering returns with the univariate models. In contrast to the regime-switching models, where the copula parameters remain static and time-variation in the dependence structure is induced by the latent Markov chain, the parameters of dynamic copulas are allowed to change in discrete time steps. The considered time-varying models are the dynamic versions of the well-known static copulas from literature, i.e. dynamic versions of both Archimedean and elliptical copulas and dynamic mixtures of these dynamic copulas. The results of the in-sample parameter estimation of the dynamic models are shown in Table 2. Independent of the model, the β coefficients are significant and show the high persistence in the correlation structures of stock returns. However, the parameters of the dynamic mixture weight process are in most cases not statistically significant. The Dynamic Student-t as well as the mixture models Gaussian/Clayton and Student-t/Clayton show the lowest AIC and BIC criteria.

Panel A of Figure 3 visualizes the evolution of the dynamic Clayton copula parameter θ_c governed by its parameter estimates listed in Table 2. The plot illustrates the adaption of the dynamic Clayton copula to the changes of the equity indices' dependence over time. As a comparison, it further shows the parameter estimate of the static Clayton as an overlaid dashed line. The parameter of the static Clayton can be seen as the average of the dynamic θ_C. The comparison with the static Clayton parameter indicates that without allowing for time-variation in the parameter, the static copula overestimates the magnitude of the parameter in most of the first half of the sample period, while underestimating it most of the time in the second half. Specifically, this means that during the unfolding of the financial crisis, the parameter of the dynamic Clayton reaches peaks in excess of 1.5 which is more than twice the magnitude of the static Clayton parameter of 0.714.[15]

[15] Furthermore, the values of both the AIC and the BIC for the dynamic Clayton copula are substantially lower than for the static version (see Internet Appendix C.2).

Table 2: Estimation Results of Multivariate Dynamic Copula Models

Copula	DC	DF	DG	DT	DFC	DGC	DTC
α			0.012	0.011		0.013	0.013
			(0.006)	(0.006)		(0.011)	(0.010)
β			0.986	0.986		0.985	0.985
			(0.129)	(0.146)		(0.195)	(0.175)
α_C	-1.199				-2.650	-4.418	-3.608
	(0.351)				(0.682)	(3.131)	(1.587)
β_C	0.917				-1.035	-0.745	-0.969
	(0.436)				(2.468)	(4.170)	(1.007)
ω_C	0.031				1.867	4.492	3.896
	(0.014)				(1.223)	(4.121)	(1.935)
α_F		-0.472			-0.074		
		(0.138)			(0.552)		
β_F		0.908			0.990		
		(0.069)			(0.284)		
ω_F		0.837			0.140		
		(0.328)			(0.513)		
α_W					-0.932	1.680	0.012
					(21.275)	(0.924)	(5.588)
β_W					-1.098	-0.249	-6.460
					(2.377)	(7.370)	(7.217)
ω_W					0.243	2.008	7.780
					(1.038)	(6.942)	(6.696)
ς				-1.435			-2.270
				(0.241)			(0.273)
φ				0.526			0.568
				(0.219)			(0.258)
$\log \mathcal{L}$	2427	2369	3476	3546	2705	3540	3581
AIC	-4848	-4732	-6948	-7084	-5392	-7064	-7142
BIC	-4833	-4717	-6938	-7063	-5346	-7023	-7090

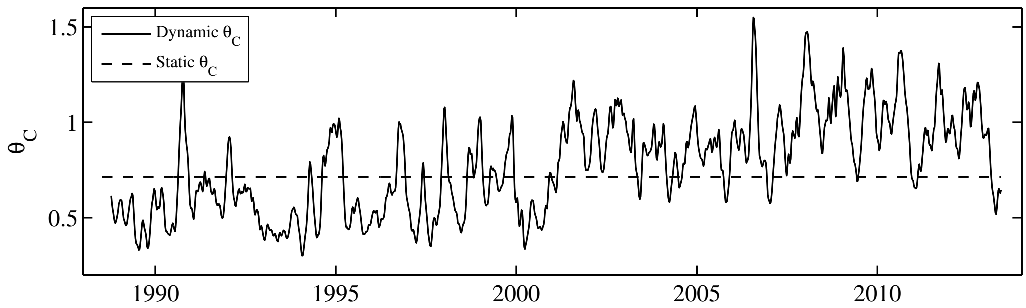
This table shows the parameters of dynamic copula parameters. Standard errors are listed in parentheses. The prefixed D stands for 'dynamic' and the copula models are abbreviated as follows: Frank (F), Clayton (C), Gaussian (G), Student-t (T), Frank & Clayton mixture (FC), Gaussian & Clayton mixture (GC), and Student-t & Clayton mixture (TC). The subscript W indicates the parameters of the dynamic mixture weight process. ς and φ are the parameters of the dynamic Student-t copula's degrees of freedom process.

Panel B of Figure 3 allows a comparison of the degrees of freedom of the dynamic Student-t copula with the ν parameter of its static counterpart. The evolution of the dynamic degrees of freedom follows a downward trend over the sample period, indicating that tail dependence has increased over time. In contrast to the findings for the Clayton copula parameter θ_C, the static degrees of freedom parameter (dashed line) is nowhere near the average of the dynamic ν as it remains below the dynamic parameter (solid line) at all times. This is consistent with Dias and Embrechts (2010), who find on the bivariate level that the degrees of freedom of the dynamic Student-t copulas is always larger than for the static Student-t copula. One may conclude that ignoring time-variation in the copula parameters might induce spurious heavier conditional tails.

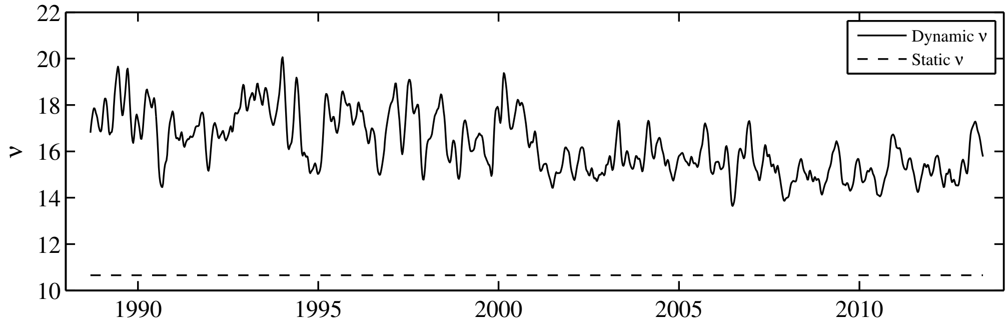
Table 3 shows the overall ranking of the in-sample model fit among the static, regime-

Figure 3: Dynamic and Static Copula Parameters

Panel A: Dynamic and Static Clayton Copula Parameters



Panel B: Dynamic and Static Student-t Copula's DOF Parameters



This figure shows in Panel A the smoothed evolution of the dynamic Clayton copula parameter (solid line) versus the static Clayton copula parameter (dashed line) over the entire sample period. Panel B shows the smoothed evolution of the dynamic Student-t copula's degrees of freedom parameter (dof) (solid line) versus the ν parameter of the static Student-t copula (dashed line) over the entire sample period.

Table 3: In-Sample Model Fit Ranking

	1.	2.	3.	4.	5.
AIC	T/GC	G/G	G/T	DTC	G/TC
BIC	DTC	DT	DGC	DG	T/GC

This table presents the top in-sample model fit rankings for the static, regime-switching, and dynamic copulas according to the Akaike (AIC) and the Bayesian (BIC) information criteria. The copulas are abbreviated as follows: Gaussian (G), Student-t (T), Clayton (C), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). The forward slash indicates the separation of the regimes in the Markov switching models while the prefixed D denotes dynamic copula models.

switching and dynamic copulas according to the two information criteria. The AIC favors the regime-switching models, while the BIC indicates that the dynamic copulas are more adequate. Compared to the dynamic copulas, the regime-switching models contain on average more parameters which entails a larger penalty for model complexity in the BIC than in the AIC. The dynamic Student-t and the dynamic Student-t-Clayton mixture stand out as they dominate the top two ranks according to the BIC.[16] The best-ranked copulas highlight that tail dependence as well as the capability of capturing asymmetries in the dependence structure are crucial features of a well-fitting model.

4 Forecast Evaluation

To test the predictive power of the different copula models, we present the methodology and results of out-of-sample backtests. We perform Monte Carlo simulations to obtain forecasted profit and loss distributions. Since the same univariate models are coupled with different copulas, differences in the return distribution forecasts are attributable to the copula functions only. The most accurate forecasts are thus produced by the model whose copula function is best capable of describing the multivariate dependence structure. The performance of the models is evaluated in a comprehensive backtesting scheme by comparing their forecasts with the observed historical portfolio returns. We use a broad range of backtesting risk measures to evaluate the predictive performance. These risk measures are *Value-at-Risk (VaR)*, *Unconditional Coverage Test* and *Independence Test* (and a *Joint Test* of both), *Basel Three-Zone Approach*, as well as *Expected Shortfall Evaluation.* We also backtest the *Entire* and *Entire Lower Tail* forecasted profit and loss distribution from the risk models.[17]

4.1 Backtesting Procedure

The backtesting procedure is based on a rolling window scheme with 520 returns. Specifically, to compute a one-week forecast of the portfolios' profit and loss distribution in time t, the univariate models and the copula functions are calibrated to the information from $t-520$ until $t-1$ with the multi-stage maximum likelihood estimation outlined in In-

[16]Further insights into the performance of Frank-Clayton mixture copula are shown in the Internet Appendix D4.
[17]A detailed description of these backtesting risk measures are provided in the Internet Appendix D.1.

ternet Appendix B. Thereafter, dependent uniform variates with the specified copula are simulated with the corresponding algorithms and subsequently transformed by inversion of the according marginal cumulative distribution function to obtain standardized residuals. These standardized residuals are then employed as the independent and identically distributed noise processes of the respective GARCH models, which reestablish the heteroscedasticity and the autocorrelation of the original returns. We thus simulated 10'000 weekly returns for each portfolio component and computed the portfolio's profit and loss. The return of the simulated equally weighted portfolio is further compared to the historical value. Taking advantage of the entire forecasted portfolio return distributions, both risk measure forecasts and density forecasts are evaluated.[18]

4.2 Overall Forecast Performance

This subsection presents the results of the backtest evaluation methods. We used as benchmark for the backtest results a multivariate normal model. Multivariate normal means that the model makes use of the same univariate GARCH processes but assumes the resulting standardized residuals to be distributed according to a multivariate normal distribution. The dependence structure between the different series is described by a static Gaussian copula.

Table 4 shows the results for the static copula models. The risk measure tests are conducted on three levels of confidence α: 90%, 95% and 99%. The hit ratios of the static models are materially different. Naturally the Basel three-zones framework classifies these models as *red*. On the three levels of α, all static mixture models' hit ratios are out of an acceptable range representing a multiple of the ones of the standalone copulas. This firstly stands in contrast to the theoretically promising setup of those models and secondly in contrast to their in-sample model fit. In fact, the static Student-t-Clayton

[18]The univariate models employed in the backtesting procedure differ from the ones outlined in Section 3. While the latter were found using the entire data set, the univariate models for the backtesting procedure are determined by choosing those specifications which reveal the lowest AIC/BIC value for the first 520 returns in the respective data set.

Table 4: Static Models Out-of-Sample Backtest Results

	α	F	C	G	T	FC	GC	TC
Hit Ratio	99%	0.078	0.019	0.026	0.026	0.186	0.185	0.187
	95%	0.121	0.101	0.097	0.097	0.250	0.258	0.258
	90%	0.169	0.178	0.165	0.165	0.295	0.299	0.303
ES Ratio	99%	0.855	0.962	0.910	0.941	0.732	0.726	0.728
	95%	0.821	1.063	0.968	0.997	0.680	0.693	0.691
	90%	0.850	1.078	1.013	1.028	0.658	0.663	0.669
Traffic Light		Red	Yellow	Yellow	Yellow	Red	Red	Red
Ind. Cov.	99%	2.328	0.589	2.702	2.702	2.778	2.344	4.098
		(0.127)	(0.443)	(0.100)	(0.100)	(0.096)	(0.126)	(0.043)
	95%	4.678	6.571	4.603	6.227	1.065	0.401	0.200
		(0.031)	(0.010)	(0.032)	(0.013)	(0.302)	(0.526)	(0.654)
	90%	2.089	4.942	2.948	2.948	0.339	0.379	0.094
		(0.148)	(0.026)	(0.086)	(0.086)	(0.560)	(0.538)	(0.759)
Unc. Cov.	99%	148.28	5.285	13.458	13.458	599.12	592.89	605.37
		(0.000)	(0.022)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	59.577	33.729	29.289	29.289	351.11	373.50	373.50
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	35.153	44.080	31.599	31.599	229.01	237.02	245.14
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	99%	150.60	5.874	16.160	16.160	601.90	595.24	609.46
		(0.000)	(0.053)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	64.255	40.301	33.892	35.516	352.17	373.90	373.70
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	37.242	49.022	34.547	34.547	229.35	237.40	245.23
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
χ^2-Test		126.76	76.345	27.349	48.076	172.43	182.79	149.24
		(0.000)	(0.000)	(0.701)	(0.026)	(0.000)	(0.000)	(0.000)
AD Test		20.297	28.984	8.248	7.652	298.13	297.89	301.21
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
KS Test		0.077	0.094	0.068	0.071	0.211	0.211	0.216
		(0.000)	(0.000)	(0.002)	(0.001)	(0.000)	(0.000)	(0.000)
Lower Tail		359.10	11.174	52.457	33.593	1325.6	1287.8	1309.5
		(0.000)	(0.011)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the static copula models. The copula models are abbreviated as follows: Frank (F), Clayton (C), Gaussian (G), Student-t (T), Frank-Clayton mixture (FC), Gaussian-Clayton mixture (GC), and Student-t-Clayton mixture (TC). α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10%-quantile) of the P&L distribution.

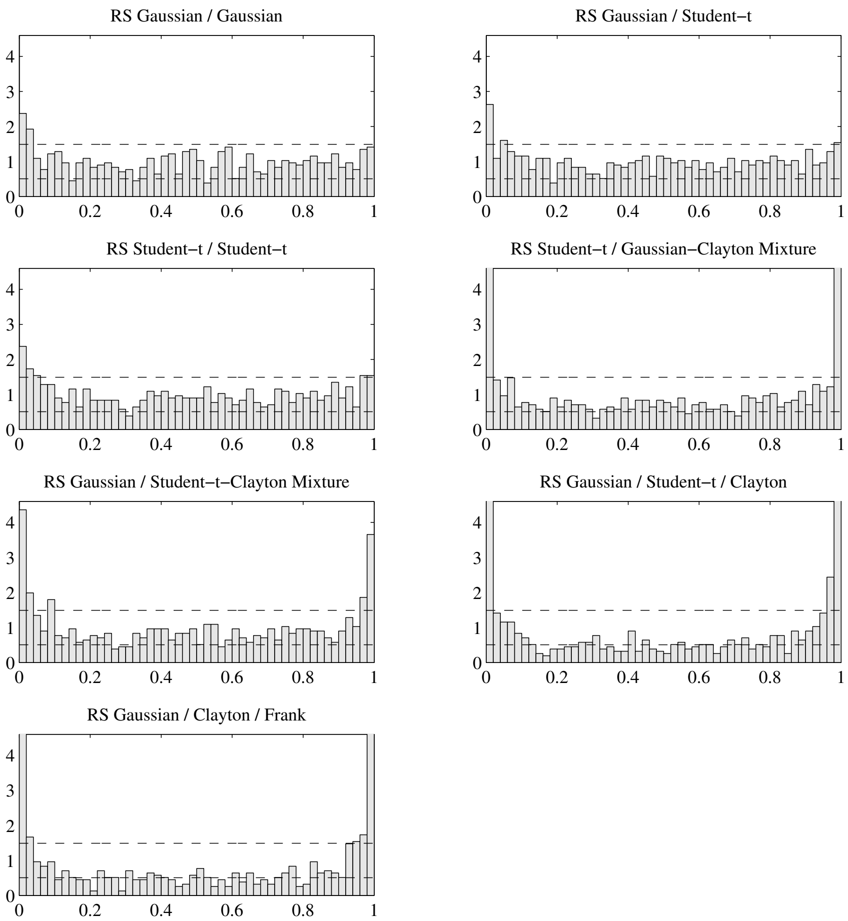
mixture and the Gaussian-Clayton mixture, which were found to have the best and third-best in-sample fit according to both information criteria, together with the Frank-Clayton mixture display the worst hit ratios among the static models. The results of the expected shortfall ratio substantiate the inferiority of the static mixture models compared to the standalone copulas. These results are consistent with the evidence for static bivariate copulas provided by Weiss (2011, p.186), who finds that mixture copulas yield "by far the worst VaR- and ES-estimates." One explanation for these results is that it is difficult to find numerically stable parameters for the static Archimedean copulas in the mixture due to the time-variation in the dependence structure.

The Frank copula is the fourth static model that is classified as *red* by the Basel traffic light system yielding an unsatisfactory hit ratio and a relatively low ES ratio. While the violations of the VaR forecasted with the Frank copula model are independent over time, the model fails the joint test due to its inaccurate unconditional coverage. The model further fails the density tests altogether owing to its incapacity to forecast the lower tail of the portfolio profit and loss distribution. The Frank copula function proves to be incapable of adequately capturing the dependence structure with its single parameter θ_F.

The Clayton, Gaussian, and Student-t copula models all reach a *yellow* traffic light classification. While the three dependence functions (just) which passes the independence coverage test for all α levels, the Clayton is the only static model to pass the unconditional coverage test and the joint test for $\alpha = 99\%$. Interestingly, the hit ratios of the Gaussian model are identical to the ones for the Student-t. The capability of the Student-t copula to capture tail dependence yields a superior prediction of the expected shortfall. The χ^2-test shows that the Gaussian copula is the most suitable static model to forecast the profit and loss distribution in its entirety. However, the forecasts are not accurate enough to pass neither the Kolmogorov-Smirnov nor the Anderson-Darling test. Comparing the 99% hit ratios of the Gaussian and the multivariate benchmark model shows the importance of incorporating fat tails and skewness in the marginal distributions, as the hit ratio of the latter model exceeds the one of the former by 1.9%.

Among the three *yellow* static models, the asymmetric Clayton copula model is most suitable to forecast the joint extreme negative returns of the portfolio constituents based on its hit ratio being closest to the expectation of 1%. Compared to the multivariate normal benchmark, the static Clayton model's forecasts for the 99% VaR are a staggering 2.6% closer to the expected hit ratio of 1%. The Clayton model's output further yields the most accurate forecasts for the entire lower tail of the portfolio return distribution as it is the only static model to pass the Lower Tail test. Note that in the in-sample analysis, the static Clayton copula yields the second poorest model fit, while the Gaussian-Clayton and the Student-t-Clayton mixture copulas were ranked top three. The backtesting results at hand show that the ranking of the in-sample model fit does not contain much information about the out-of-sample predictive power.

Figure 4: Backtest PIT Histograms of Regime-Switching Copula Models



This figure shows relative frequency of occurrence of the probability integral transforms of the portfolio returns taken with respect to the regime-switching (RS) copula models' forecasted probability density distribution. The histogram is segmented into 50 bins of equal size. The overlaid dashed lines indicate the 95%-confidence interval for the heights of the individual bins under the null hypothesis that the probability integral transforms are $(0, 1)$ uniform distributed.

The results of the regime-switching models, shown in the Internet Appendix E4., suggest that the two-regime models G/G, G/T, and T/T fare considerably better, all attaining a yellow classification compared to the mixture models G/TC and T/GC, as well as the three-regime models, all of which are classified as red. However, with hit ratios being substantially larger than the expected value, they fail the unconditional coverage and the joint test at all confidence levels.[19] While in particular the G/G regime-switching model seems to produce acceptably accurate forecasts for the center of the profit and loss distribution based on both χ^2- and KS-test results, all regime-switching models fail the Lower Tail test. The best fitting model according to the in-sample analysis (T/GC) produces some of the worst profit and loss distribution forecasts among the regime-switching specifications. However, ranks two and three according to the in-sample information criteria are attributed to the G/G and the G/T copulas, which in fact belong to the regime-switching models with the best out-of-sample forecast power.

Figures 4 and 5 show the histograms of the normalized frequency of probability integral transformed portfolio returns, which are divided into 50 bins of equal size. The overlaid dashed lines indicate the indicate the 95% confidence level. Note that if a model's forecasts of the profit and loss distributions were accurate, all bars in a histogram would be equally high at exactly one. Figure 4 indicates that two out of five regime-switching copula models have difficulties to forecast both tails of the portfolio profit and loss distribution. Both structures where one regime is modeled by a mixture copula display large bars at the lowest and highest quantiles of the histograms. The two models yield enormous hit ratios, low ES ratios, classify as red according to the Basel regulatory framework, and fail the unconditional coverage and the joint tests on all levels of alpha. The static mixtures prove to be unsuccessful in forecasting also as a part of the regime-switching setup.

A comparison of the backtesting results of the Gaussian/Gaussian regime-switching model with those of the static Gaussian copula shows that the former does produce slightly more accurate forecasts in terms of hit ratio and expected shortfall ratio on all levels of α. The improved results for the Lower Tail test further indicate that the G/G issues better forecasts than the static Gaussian. Note that the regime-switching Gaussian copula is the

[19] Adding a third regime does not improve the forecast performance. On the contrary, the G/T/C yields significantly worse results than the G/T as shown in the Internet Appendix E4.

only model among both static and regime-switching structures to pass the Kolmogorov-Smirnov test. According to the test results of the χ^2-test, the T/T regime-switching copula yields better profit and loss density forecasts compared to the static Student-t copula. However, none of the models passes the unconditional coverage or joint test, and the hit ratios as well as the ES ratios are virtually the same. This result is underpinned by an even worse Lower Tail test statistic of the Markov switching setup.

The backtest results for the dynamic models in Table 5 indicate that the dynamic Frank copula is better in nearly all backtest measures compared to its static counterpart. However, they are far from satisfactory, which means that the standalone Frank copula is neither in static nor in dynamic form a suitable model for the equity index returns. The histogram of the dynamic Frank copula's results in Figure 5 illustrates that the incapability to capture the dependencies of the lower extreme returns is the cause of the dynamic Frank copula's failure. This is substantiated by its failure to pass any of the density tests.

With a substantially higher bar for the highest 2%-quantile in the histogram, the dynamic Clayton is not a good model to forecast the upper part of the portfolio return distribution. The strengths of the dynamic Clayton model lie in its capability to capture lower tail dependence. This is confirmed by the best forecasts for the lower tail of the equity portfolio's return distribution showing the highest p-value for the Lower Tail test among all static, regime-switching, and dynamic copula models. The dynamic version of the Clayton further provides lower hit ratios than the static Clayton for all tested α; in fact, it yields the overall lowest hit ratio on $\alpha = 99\%$. The dynamic Clayton is furthermore the only model to qualify as *green* in the Basel three-zones approach.

Note that all dynamic mixture copulas yield substantially better results compared to their static counterparts. While the static mixtures proved to be virtually useless in terms of forecast power, the dynamic Gaussian-Clayton mixture and the Dynamic Student-t-Clayton mixture classify as *yellow*. The dynamic Student-t-Clayton mixture copula even

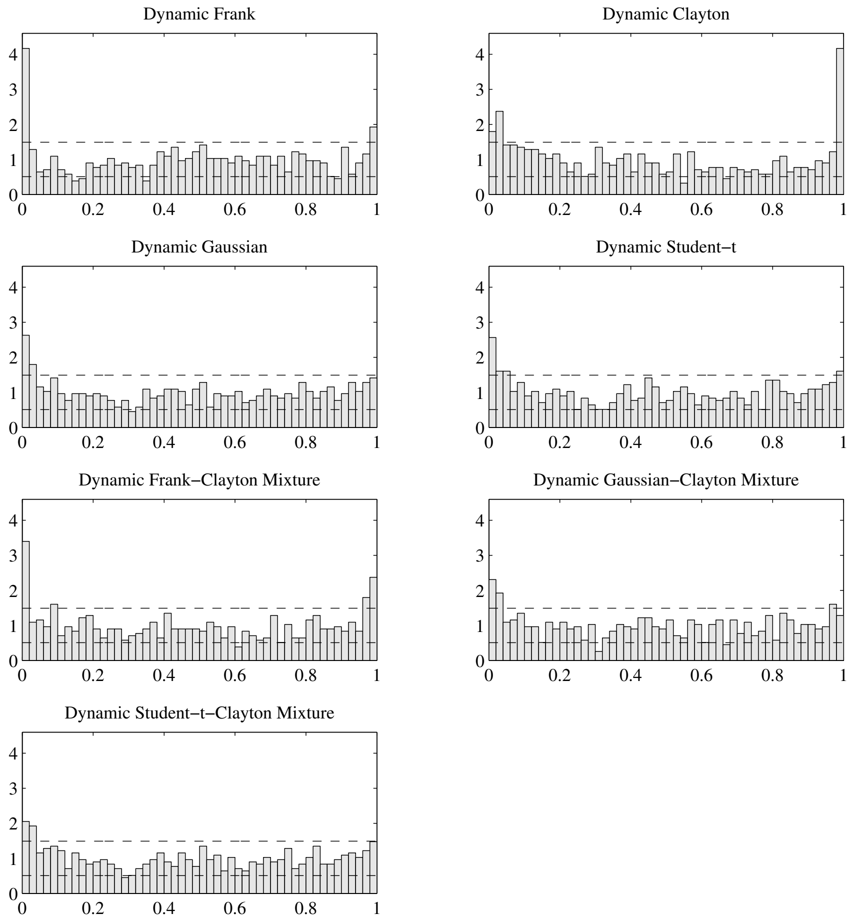
Table 5: Dynamic Models Out-of-Sample Backtest Results

	α	DF	DC	DG	DT	DFC	DGC	DTC
Hit Ratio	99%	0.072	0.015	0.026	0.024	0.041	0.023	0.017
	95%	0.114	0.099	0.097	0.095	0.100	0.095	0.096
	90%	0.158	0.164	0.160	0.162	0.164	0.156	0.155
ES Ratio	99%	0.850	0.928	0.915	0.950	0.977	1.356	0.877
	95%	0.828	1.086	0.982	0.972	0.924	1.028	1.021
	90%	0.854	1.070	1.001	1.003	0.970	1.025	1.026
Traffic Light		Red	Green	Yellow	Yellow	Red	Yellow	Yellow
Ind. Cov.	99%	2.286	1.862	6.290	0.497	7.203	0.626	0.441
		(0.131)	(0.172)	(0.012)	(0.481)	(0.007)	(0.429)	(0.507)
	95%	3.959	2.894	3.200	5.406	3.871	1.492	3.522
		(0.047)	(0.089)	(0.074)	(0.020)	(0.049)	(0.222)	(0.061)
	90%	5.131	3.269	4.333	2.223	1.720	4.432	2.843
		(0.024)	(0.071)	(0.037)	(0.136)	(0.190)	(0.035)	(0.092)
Unc. Cov.	99%	127.451	1.962	13.458	11.595	42.710	9.840	2.917
		(0.000)	(0.161)	(0.000)	(0.001)	(0.000)	(0.002)	(0.088)
	95%	50.316	30.740	29.289	26.474	32.221	26.474	27.867
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	24.995	30.451	27.121	28.212	30.451	23.962	22.947
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	99%	129.737	3.824	19.748	12.092	49.913	10.466	3.358
		(0.000)	(0.148)	(0.000)	(0.002)	(0.000)	(0.005)	(0.187)
	95%	54.274	33.635	32.489	31.881	36.091	27.966	31.389
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	30.126	33.720	31.454	30.435	32.171	28.394	25.790
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
χ^2-Test		123.620	67.379	33.151	37.737	35.672	26.825	30.809
		(0.000)	(0.001)	(0.411)	(0.223)	(0.390)	(0.632)	(0.425)
AD Test		16.561	24.169	7.239	7.928	13.480	5.935	6.319
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.001)
KS Test		0.071	0.088	0.066	0.065	0.065	0.058	0.064
		(0.001)	(0.000)	(0.002)	(0.003)	(0.003)	(0.010)	(0.003)
Lower Tail		331.268	8.352	39.353	34.571	107.550	18.414	18.510
		(0.000)	(0.039)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the dynamic copula models. The models are abbreviated as follows: Dynamic (D), Frank (F), Clayton (C), Gaussian (G), Student-t (T), Frank-Clayton mixture (FC), Gaussian-Clayton mixture (GC), and Student-t-Clayton mixture (TC). α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10%-quantile) of the P&L distribution.

yields the second-best hit ratio for the equity data on $\alpha = 99\%$ and passes the joint test on this level of significance. This shows the fundamental importance of accounting for time-variation in mixture copula models. In the in-sample analysis of the dynamic models, the dynamic Student-t-Clayton mixture stood out with the lowest AIC and BIC values indicating a good fit which is underpinned by the results of the backtesting procedure. The dynamic Clayton copula, however, has the second poorest in-sample fit among the dynamic models but yields the best forecasts of the lower tail of the equity portfolio's

Figure 5: Backtest PIT Histograms of Dynamic Copula Models



Relative frequency of occurrence of the probability integral transforms of the portfolio returns using the dynamic copula models' forecasted probability density distribution. The histogram is segmented into 50 bins of equal size. The overlaid dashed lines indicate the 95% confidence interval for the heights of the individual bins under the null hypothesis that the probability integral transforms are $(0, 1)$ uniform distributed.

profit and loss distribution. This shows that the in-sample model fit criteria might be too focused on the center of the distribution, whereas the risks are located in the lower tail.

4.3 Crisis Forecast Performance

The standard risk management models employed by the financial industry have drawn wide criticism for their performance during the financial crisis (see, e.g., Skoglund, Chen, and Erdman (2010); Das, Embrechts, and Fasen (2013)). Many financial institutions reported a sharp increase in the number of VaR violations in the unfolding of the crisis, which underpinned the perception that risk forecasting models perform well except in times of crises. While in the previous section, the scope of the backtest covered the entire time frame evaluating out-of-sample forecasts from 1998 until 2013, this subsection concentrates on the models' performance during the last financial crisis and the European sovereign debt crises. To ensure comparability, we use the same backtest procedures including the same univariate model specifications as in Subsection 4.2. The performance of the models is evaluated following the Basel supervisory framework (Basel Committee on Banking Supervision (2013)), which demands backtests of the risk model based on the VaR measure on the 99% confidence level.

Firstly, the crisis forecast performance analysis focuses on the out-of-sample accuracy of the risk forecasts of the different models during the crisis period from January 7, 2007 until January 5, 2011. The time period therewith covers the unfolding of the financial turmoil from the disruptions in the subprime mortgage market to the virulent global financial crisis as well as the subsequent European debt crisis. Secondly, the changes in VaR violations of the presented models in reaction to the outbreak of the crises are investigated over time.

Table 6 summarizes the results of the backtests for all presented copula models during the crisis period. The therein reported measures refer to the 99%-confidence level with the exception of the Lower Tail test, which assesses the model's capability to forecast the density of the profit and loss distribution below the 10%-quantile. According to the Basel three-zones approach, the Frank copula model does not produce acceptable forecasts during the crisis neither in its dynamic nor in its static form. Furthermore, all static mixture copulas, the dynamic Frank-Clayton mixture, the three-state regime-switching models and the two-state models with mixture regimes are also labeled *red*, which means that the accuracy of their VaR forecasts is not acceptable either. The inaccuracy of these models' forecasts is substantiated by their failure to pass the unconditional coverage and the joint test as well as the Lower Tail test. The Dynamic-Clayton copula is still the superior model, as it is classified as “Green”, which confirms the results from the in-sample analysis.

Table 6: Financial Crisis Out-of-sample Backtest Results (for $\alpha = 99\%$)

Static	F	C	G	T	FC	GC	TC
Hit Ratio	0.077	0.019	0.029	0.033	0.201	0.211	0.206
ES Ratio	0.811	0.915	0.909	0.997	0.702	0.722	0.720
Traffic Light	Red	Green	Yellow	Yellow	Red	Red	Red
Ind. Cov.	0.025	0.157	0.356	1.545	3.866	2.515	4.721
	(0.875)	(0.692)	(0.550)	(0.214)	(0.049)	(0.113)	(0.030)
Unc. Cov.	38.270	1.391	4.910	7.220	180.470	193.447	186.929
	(0.000)	(0.238)	(0.027)	(0.007)	(0.000)	(0.000)	(0.000)
Joint Test	38.295	1.548	5.266	8.765	184.336	195.961	191.650
	(0.000)	(0.461)	(0.072)	(0.012)	(0.000)	(0.000)	(0.000)
Lower Tail	133.466	6.872	16.678	9.537	446.127	394.398	413.505
	(0.000)	(0.076)	(0.001)	(0.023)	(0.000)	(0.000)	(0.000)
Regime-Switching	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
Hit Ratio	0.029	0.029	0.029	0.196	0.191	0.048	0.096
ES Ratio	0.964	0.925	0.943	0.667	0.679	0.804	0.814
Traffic Light	Yellow	Yellow	Yellow	Red	Red	Red	Red
Ind. Cov.	0.356	2.104	0.296	3.059	3.757	3.189	16.435
	(0.550)	(0.147)	(0.586)	(0.080)	(0.053)	(0.074)	(0.000)
Unc. Cov.	4.910	4.910	4.910	174.070	167.731	15.795	56.120
	(0.027)	(0.027)	(0.027)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	5.266	7.014	5.206	177.129	171.488	18.984	72.555
	(0.072)	(0.030)	(0.074)	(0.000)	(0.000)	(0.000)	(0.000)
Lower Tail	11.819	9.272	14.274	480.834	408.501	52.227	113.822
	(0.008)	(0.026)	(0.003)	(0.000)	(0.000)	(0.000)	(0.000)
Dynamic	DF	DC	DG	DT	DFC	DGC	DTC
Hit Ratio	0.081	0.010	0.029	0.029	0.048	0.029	0.014
ES Ratio	0.855	0.769	0.935	0.983	1.009	0.996	0.818
Traffic Light	Red	Green	Yellow	Yellow	Red	Yellow	Green
Ind. Cov.	0.141	0.039	2.104	0.356	0.485	0.356	0.088
	(0.707)	(0.844)	(0.147)	(0.550)	(0.486)	(0.550)	(0.767)
Unc. Cov.	42.547	0.004	4.910	4.910	15.795	4.910	0.353
	(0.000)	(0.950)	(0.027)	(0.027)	(0.000)	(0.027)	(0.553)
Joint Test	42.688	0.043	7.014	5.266	16.280	5.266	0.441
	(0.000)	(0.979)	(0.030)	(0.072)	(0.000)	(0.072)	(0.802)
Lower Tail	118.078	4.043	9.350	11.086	36.622	6.590	6.376
	(0.000)	(0.257)	(0.025)	(0.011)	(0.000)	(0.086)	(0.095)

This table reports the backtest evaluation results for the forecasting period from January 2007 until January 2011 for all copula models. The copula models are abbreviated as follows: Gaussian (G), Student-t (T), Frank (F), Clayton (C), Frank-Clayton mixture (FC), Gaussian-Clayton mixture (GC), and Student-t-Clayton mixture (TC). The prefixed D denotes dynamic copulas. The confidence level of VaR is $\alpha = 99\%$. The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns when VaR(α) is violated corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the test for conditional coverage. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the lowest decile of the P&L distribution. P-values are given in parentheses.

Eight models qualify as *yellow*, among which seven yield the identical hit ratio of 2.9%: the Gaussian copula in the static and dynamic version, the regime-switching models G/G, G/T and T/T, the dynamic Student-t copula and the dynamic Gaussian-Clayton mixture. These seven models' VaR forecasts are violated almost 50% less than those of the multivariate normal benchmark. While they are identical with regards to their VaR forecasts, their expected shortfall ratios reveal the differences in the accuracy of the average forecasted loss given a VaR violation: the least accurate ES ratio among the *yellow* models is found with the static Gaussian copula and the closest to the expected value of 1 is the ES ratio of the dynamic Gaussian-Clayton mixture followed by the dynamic Student-t copula. Note that the direct comparisons of the expected shortfall ratios is expressive in this case since the compared models all have the same hit ratio. As the ES ratio hinges on the hit ratio, a sole comparison of the former without considering the latter is not very meaningful.

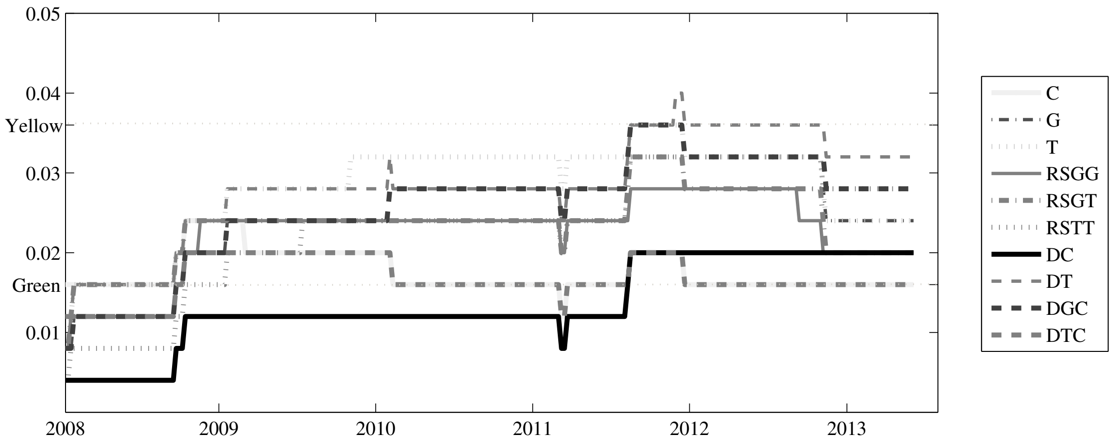
Three copulas, however, are distinctly more accurate during the financial crisis than all other models as they are the only ones to score *green* according to the Basel framework: the static Clayton copula, the dynamic Clayton copula, and the dynamic Student-t-Clayton mixture. Their results for the coverage tests and the joint test confirm their superiority, which is further substantiated by their results for the Lower Tail test. The dynamic Clayton copula clearly produces the most accurate VaR forecasts during the financial crisis with a hit ratio exactly equal to the theoretical expectation of 1%. The highest p-value for the Lower Tail test confirms the dynamic Clayton's superior capabilities to forecast the lowest decile of the equity portfolio's profit and loss distribution. While the results for the static Clayton highlight the importance of modeling lower tail dependence in a multivariate setting, the substantially increased forecast accuracy of the dynamic Clayton copula emphasizes the significance of also modeling time-variation in the dependence structure of the equity index portfolio.

To investigate the models' reaction to the unfolding of the financial crisis and European debt crisis, Figure 6 visualizes the hit ratios over time. The plot shows the hit ratios at the 99% level, calculated on a rolling window basis using the preceding 250 returns in allusion to the Basel regulatory framework (Basel Committee on Banking Supervision

(2013)). The graph includes the hit ratios of those models which qualify as *green* at the beginning of 2008. Naturally, the hit ratio of an ideal model would remain close to the expected hit ratio of 0.01 showing little to no reaction to the outbreak of the crisis. The three zones of the Basel framework are marked on the y-axis of the plot: models with a hit ratio in time t below the mark "Green" respectively "Yellow" classify accordingly, while models whose hit ratios in time t exceed "Yellow" are ranked *red*. All depicted models remain in the *green* zone until September 2008, when the hit ratios increase sharply in reaction to the bankruptcy of the investment bank Lehman Brothers and the collapse of the insurance firm AIG. With the exception of the dynamic Clayton copula, all depicted models turn from green to yellow in only a few weeks. The dynamic Clayton's hit ratio, however, shows the smallest reaction to the outbreak of the global crisis preserving its *green* status until the aggravation of the Eurozone crisis in mid-2011. At the same time, the static Gaussian and the static Student-t show the strongest immediate reaction to the eruption of the crisis. The second and third best model according to the results in Table 6 are easily identifiable in Figure 6 as together with the dynamic Clayton their hit ratios are closest to the expected value of 0.01: the dynamic Student-t-Clayton mixture and the static Clayton. Whilst the dynamic Clayton model provides the most accurate forecasts during the global financial crisis remaining *green* until mid 2011 the other two models cope better with the aftermath of the Eurozone crisis.

Table 7 relates the results of the models for the financial crisis period (upper panel) to the overall performance of the models documented in the previous Subsection 4.2 (lower panel), by listing the top rankings of the models for all portfolios according to the accuracy of their VaR(99%) forecasts. The comparison shows, that the models' ranking during the financial crisis is largely consistent with the overall ranking. The Clayton copula stands out, as it is ranked first. In general, the table demonstrates that the dynamic copulas yield VaR forecasts of superior accuracy compared to both the regime-switching and the static copulas. It further displays that asymmetric dynamic copulas generally dominate

Figure 6: Hit Ratio Evolution during the 2007/08 Financial and European Debt Crisis



The plot shows the evolution of the hit ratios of several copula models starting from the outbreak of the financial crisis. Depicted are the hit ratios at the 99% level over a rolling window of 250 returns of all those models which classified as "Green" by the Basel II framework at the beginning of 2008. Hit ratios below the dotted line labeled "Green" respectively "Yellow" are classified accordingly by the Basel regulatory framework. The models with ratios above the line "Yellow" are categorized as "Red" according to the Basel traffic light approach.

the top five rankings of the crisis and the overall performance.

Table 7: Out-of-sample Forecast Accuracy Ranking

	1.	2.	3.	4.	5.
Financial Crisis	DC	DTC	C	DGC	DT
Overall	DC	DTC	C	DGC	DT

This table presents the top rankings of the out-of-sample VaR(99%) forecast accuracy for the static, regime-switching, and dynamic copulas. The rankings for models with identical hit ratios are determined by the accuracy of their ES ratio. The upper panel refers to the financial crisis out-of-sample performance (see Subsection 4.3) while the lower panel refers to the overall out-of-sample performance of the models (see Subsection 4.2). The copulas are abbreviated as follows: Gaussian (G), Student-t (T), Clayton (C), Frank-Clayton mixture (FC), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). The prefixed D denotes dynamic copula models.

5 Conclusion

A proper specification of financial assets' multivariate distribution is essential to forecast the risk of a portfolio. This study investigated the importance of modeling time-variation and asymmetries in the dependence structure to forecast the risk of a portfolio of international equity indices. The level of dependence as well as the amount of lower tail dependence was found to vary substantially over the sample period. As a first approach to model this time-variation in the interdependence of the portfolio constituents, regime-switching copula models were introduced. To enhance the flexibility of this set-up, multivariate mixture copulas were employed to characterize states. As a second approach to account for time-varying dependencies, dynamic copula models were presented. Enhancing the dynamic Student-t copula's capability to adapt both level of dependence and strength of tail dependence, its degrees of freedom parameter was also allowed to vary over time. Finally, the dynamic copulas were combined into dynamic mixture structures.

The in-sample comparison of the time-varying copulas with their static counterparts exhibited that the best fitting copulas are dynamic models which allow for tail dependence. In particular, the dynamic Student-t copula and the dynamic Student-t-Clayton mixture copula were found to have a superior in-sample fit. The best-ranked copulas indicate the importance of accounting for time-variation, highlight that tail dependence is a crucial feature of a well-fitting model and that the capability of capturing asymmetries in the dependence structure yields the winning edge.

A comprehensive backtest of the models' out-of-sample portfolio return forecasts was conducted. The results revealed that the choice of copula model has a large impact on the forecast accuracy of the portfolio profit and loss distribution's lowest decile. The differences are particularly pronounced for risk measures on the 99% confidence level. Among the static models, the Clayton copula yields by far the best results, while the symmetric Frank copula model with asymptotically independent tails was found to be inappropriate to characterize the portfolios' dependence structure. This underpins the importance of modeling asymmetries and accounting for lower tail dependence.

The comparison of the static Gaussian with the multivariate normal benchmark model showed that allowing for fat tails and asymmetries in the marginal models improves the forecast accuracy. Failing to model time-variation in the multivariate mixture copulas results in entirely inaccurate forecasts. Accounting for time-varying dependencies by employing the dynamic versions of the Student-t-Clayton mixture and the Gaussian-

Clayton mixture copulas, however, yields the second and third best VaR(99%) forecasts. Evidence from the in-sample analysis suggests that the crucial part is to account for time-variation in the mixture's copula parameters, while a static mixture weight might be sufficient.

Accounting for time-variation by switching between different static copulas as in the regime-switching models led to an improvement of some models' forecasts. Particularly, allowing for a second Gaussian state as in the Gaussian/Gaussian regime-switching copula improves the forecast accuracy on almost all VaR confidence levels. Furthermore, the density test results indicate an increase in accuracy of the entire forecasted return distribution when allowing for two instead of only one Gaussian copula regime. This evidence suggests, that multivariate regime-switching copulas are best set up with two states each consisting of one standalone copula.

The dynamic copulas have generally been found to produce forecasts of superior accuracy compared to both static and regime-switching copula models. Specifically, the predictive power of the dynamic Student-t and all dynamic mixture copula models is stronger compared to their static versions on all three significance levels. The best forecasts for the lower tail of the profit and loss distribution are produced by the dynamic Clayton copula model, which highlights the importance of modeling both positive lower tail dependence and time-variation in the dependence structure. This stands in contrast to the in-sample model fit results, where the dynamic Clayton copula was ranked in the rearmost positions. Generally, the goodness of in-sample fit was not found to provide dependable indications of predictive power. However, the in-sample ranking did point to the superiority of the time-varying models compared to the static copulas.

Finally, the tests of models' forecast accuracy for the financial and European debt crisis underpin the superiority of the Clayton copula model. The multivariate dynamic Clayton copula accurately forecast the VaR(99%) with a hit ratio of 1% followed by the dynamic Student-t-Clayton mixture copula with 1.4%. The employment of the dynamic

Clayton copula model instead of the multivariate normal model would have scaled down the number of VaR(99%) violations during the financial crisis by more than 82%.

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