Multiple Risk-Sharing: An Analysis of the Policyholder’s Preference

Lukas Reichel, Hato Schmeiser and Florian Schreiber
Institute of Insurance Economics
University of St.Gallen

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Motivation: Insurance Demand under Default Risk

Bilateral insurance contract under default risk

Policyholder \rightarrow \text{Premium} \rightarrow \text{Indemnification} \rightarrow \text{Insurer 1}

Indemnification is subject to a default risk of the insurer

What is the optimal coverage level $\alpha^*$ if the insurer’s ruin probability is non-zero?

Doherty & Schlesinger (1990):

- Insurance demand is affected by default risk; classic Mossin-theorem does not hold anymore
- Given actuarial fair premium, no unambiguous results whether over- or under-insurance is optimal

Mahul & Wright (2007):

- Given actuarial fair premium, over-insurance is optimal iff the insurer’s recovery rate is above a threshold
Motivation: Insurance Demand under Default Risk

Multiple risk-sharing in, e.g., industrial insurance, reinsurance,...

Policyholder

Premium

Indemnification

Insurer 1
Insurer 2

Insurer 3
Insurer 4

Indemnification is subject to a default risk of the insurers

Pro rata sharing of risk and premium

Question 1: How is the insurance demand affected if default risk can be diversified by multiple risk-sharing?

Insurance-Demand-Curve

Insurance Coverage

Price for Insurance

increase

reduction

?
Motivation: Insurance Demand under Default Risk

Multiple risk-sharing in, e.g., industrial insurance, reinsurance,…

Policyholder

Insurer 1  Insurer 2  Insurer 3  Insurer 4

Premium

Indemnification

Indemnification is subject to a default risk of the insurers

Pro rata sharing of risk and premium

Question 2: How is the insurance demand affected if the fee for a risk-management measure increases?

Insurance-Demand-Curve

Price for Insurance

Insurance Coverage

increase

reduction

Basic Insurance Coverage
(multiple with default risk)

Insurer 1  Insurer 2

Premium

Indemnification

Insurer 3  Insurer 4

Pro rata sharing of risk and premium

Fee

Default-Free Risk-Management Measure (RMM)

Provides replacing payment if one/several (re)insurers fail

Examples: Letter of Credit, Guarantees, Credit Default Swap, Guarantee Fund, Deposits
Model I: Insurance Coverage and External RMM

**Default-Free Risk-Management Measure (RMM)**

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**Basic Insurance Coverage** (multiple with default risk)

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Pro rata sharing of risk and premium

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Assuming that the number of insurers $n$ is exogenously given, the policyholder decides on two variables:

- The quantity of insurance coverage denoted by $e_I$ (at the premium of $c_I(e_I)$)
- The quantity of the risk-management measure denoted by $e_R$ (at the fee of $c_R(e_R)$)

The policyholder can take three positions at the total costs of $c_I(e_I) + c_R(e_R)$:

- $e_I = e_R$ means a perfect hedging $\rightarrow$ the policyholder eliminates the default risk entirely
- $e_I > e_R$ means an under-hedging $\rightarrow$ policyholder accepts a remaining default risk
- $e_I < e_R$ means an over-hedging $\rightarrow$ policyholder bets on the default of (re)insurers
Model II: Wealth States and Utility

(1) Binary loss event: 0 with probability $1 - p$, $l > 0$ with probability $p$
(2) Multiple risk-sharing: insurer $i = 1, \ldots, n$ gets $\frac{1}{n} c_i(e_l)$ of the premium and pays $\frac{1}{n} e_l$ in a loss event
(3) The insurers’ defaults are assumed to be stochastically independent
(4) Each insurer fails to compensate its share with probability $q$ (ruin probability = $qp$, i.i.d-assumption)
(5) The insurer’s default-to-liability ratio equals $0 \leq \tau \leq 1$, i.e., $(1 - \tau) \frac{1}{n} e_l$ are still paid in a default event
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The policyholder possesses a vNM utility function $u$ with $u' > 0$ and $u'' < 0$. The policyholder’s objective:

$$
\max_{e_l, e_R} U_n = \max_{e_l, e_R} \left\{ (1 - p)u(W_{NL}) + p \sum_{k=0}^{n} \binom{n}{k} q^k (1 - q)^{n-k} u(W_{L,k}) \right\}
$$
Model III: Premium/Fee Principle

Expected value calculus:

Payoff from insurance contract: \( E[I_n] = e_I p (1 - \tau q) \),
Payoff from risk-management measure: \( E[R_n] = e_R p q \tau \).

Expected payoffs do not depend on \( n \)

But: \( Var[I_n] > e_I^2 p (1 - p) \) as \( n \to \infty \) (=variance of a default-free insurance policy)

Assumed premium/fee principle: Actuarial Fair Premium/Fee x Proportional Cost Loading

Premium for insurance coverage: \( c_I(e_I) = E[I_n](1 + \lambda_I) = (e_I p - e_I p q \tau )(1 + \lambda_I) =: e_I \pi_I(1 + \lambda_I) \),
Fee for risk-management measure: \( c_R(e_R) = E[R_n](1 + \lambda_R) = e_R p q \tau (1 + \lambda_R) =: e_R \pi_R(1 + \lambda_R) \).

- By assumption, \( \lambda_I \) is non-dependent on \( n \)
- Assumption might be unmet (especially for large \( n \)) due to economies of scale (Mayers & Smith, 1981)
- Premium, fee and expected wealth do not depend on \( n \)
  \( \Rightarrow \) Multiple risk-sharing has a mean-preserving effect
  \( \Rightarrow \) Under this premium principle SOC for the maximization problem \( \max_{e_I, e_R} U_n \) is fulfilled
At first, \( e_R = 0 \) (no risk-management measure at hand)

*Policyholder’s utility*

\[
U_n = (1 - p)u(W_{NL}) + p \sum_{k=0}^{n-1} \binom{n}{k} q^k (1 - q)^{n-k} u(W_{L,k})
\]

*can be interpreted as Bernstein polynomial. Thereby, one can conclude:*

i. \( U_{n+1} > U_n \) for all \( n \geq 1 \), i.e., \( U_{n+1}(e_{n+1}^*) > U_n(e_n^*) \),

ii. \( \lim_{n \to \infty} U_n = (1 - p)u(W_{NL}) + pu(w - c_i(e_i) - c_R(e_R) - l + e_i(1 - qr)) \),

iii. \( \lim_{n \to \infty} e_{l,n}^* = \frac{e_{l,ND}^*}{(1-qr)} \).

- Example for actuarial fair premiums (\( \lambda_l = 0 \)): \( \lim_{n \to \infty} e_{l,n}^* = \frac{l}{(1-qr)} \); i.e., over-insurance is optimal
- Utility is monotonously increasing in \( n \); is the sequence \( e_{l,1}^*, e_{l,2}^*, e_{l,3}^*, ... \) monotonously increasing, too?
Numeric Example: Multiple Risk-Sharing

Counter-Example:

\[ u(x) = -\exp(-\beta x). \]

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<tr>
<th>Initial wealth ( w )</th>
<th>Loss probability ( p )</th>
<th>Loss size ( l )</th>
<th>Cost factor ( \lambda_l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.05</td>
<td>1.0</td>
<td>0.0</td>
</tr>
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</table>

- No unambiguous monotonicity for the optimal insurance quantity
- High risk-aversion rather results in a decreasing sequence \( e_{1,n}^* \)
- For \( \tau \) close to 1, high risk-aversion implies non-monotonicity for \( e_{1,n}^* \)
- Fast convergence to \( \frac{1}{(1-\tau)} \)
- \( q \) does not change results qualitatively but acts as scaling factor
Now, $e_R \neq 0$ (risk-management measure accessible)

Given the aforementioned model, the policyholder will prefer a perfect hedging (i.e. $e_{I,n}^{*} = e_{R,n}^{*}$), iff

$\lambda_I = \lambda_R$.

In this case, the optimal quantity equals the optimal quantity of a default-free insurance contract.

$\rightarrow$ For $\lambda_I \neq \lambda_R$, policyholder prefers an under(over)-hedged position.
Results III: Demand Effects of RMM

i. Given CARA-preference, an increasing cost loading for the risk-management measure results in a decreasing demand for the risk-management measure, i.e.,

\[ \frac{\partial e_{R,n}^*}{\partial \lambda_R} < 0. \]

ii. Given CARA-preference, an increasing cost loading for the risk-management measure results in an increasing demand for the insurance coverage (i.e., \( \partial e_{I,n}^* / \partial \lambda_R > 0 \)) iff

\[ \sum_{k=0}^{n} \binom{n}{k} q^k (1 - q)^{n-k} \frac{k}{n} \left( 1 - \pi_I - \frac{k}{n} \right) u''(W_{L,k}) < 0. \]

A sufficient condition is given by \( \tau \leq (1 - p(1 + \lambda_I))/(1 - pq(1 + \lambda_I)) \).

iii. Given \( e_{I,n}^* = e_{R,n}^* \) and CARA-preference, an increasing cost loading for the risk-management measure results in an increasing demand for the insurance coverage (i.e., \( \partial e_{I,n}^* / \partial \lambda_R > 0 \)) iff

\[ n \geq \frac{1-q}{(1-\pi_I)\tau^{-1} - q}. \]

Note: \( 1 \geq \frac{1-q}{(1-\pi_I)\tau^{-1} - q} \iff \tau \leq (1 - p(1 + \lambda_I))/(1 - pq(1 + \lambda_I)). \)
Numeric Example: Demand Effect

Counter-Example:

\[ u(x) = -\exp(-\beta x). \]

<table>
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<tr>
<th>Initial wealth ( w )</th>
<th>Loss probability ( p )</th>
<th>Default probability ( q )</th>
<th>Loss size ( l )</th>
<th>Risk-aversion ( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.05</td>
<td>0.1</td>
<td>1.0</td>
<td>3.0</td>
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- For \( n = 1 \) the insurance-demand curve is shifted to the left if the RMM becomes more expensive (less insurance is demanded).
- For \( n = 2 \) the insurance-demand curve is shifted to the right if the RMM becomes more expensive (more insurance is demanded).
Summary:

• In our framework, multiple risk-sharing is a costless measure to diversify default risk

• Better does not always mean more: despite increasing utility, non-monotonous relationship between number of insurers and demanded insurance quantity in multiple risk-sharing possible

• Demand on available risk-management measure depends on difference of cost-loading factors $\lambda_I$ and $\lambda_R$
  \[ \rightarrow \text{Three possible states: perfect hedging, over-hedging, under-hedging} \]

• Given CARA, insurance coverage serves as substitute for the risk-management measure if default rate is below a threshold

• Multiple risk-sharing lowers this threshold

Open issues:

• Analysis on DARA & IARA

• Easing i.i.d-assumption on insurer default; other multiple risk-sharing apart from quota share
Thank you

Lukas Reichel
Institute of Insurance Economics
University of St. Gallen
Tannenstrasse 19
9000 St. Gallen
Switzerland
lukas.reichel@unisg.ch