A Monetary Model of the Eurozone

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Abstract

This technical appendix documents a three region model of the world economy. Italy is represented in detail, the rest of the Eurozone is stylized, and the rest of the world is largely exogenous. The model includes wage rigidity, endogenous sovereign bond prices, balances sheet constraints and endogenous leverage of firms and banks and sovereign debt dynamics with fiscal consolidation policies. It allows for alternative monetary regimes with autonomous and common monetary policy.

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1 Model of a EZ Country

The model accounts for three regions with separate monetary policy in Italy and rest of EZ. The RoW good is the numeraire. Two exchange rates are endogenous. Italy is modeled in detail, including wage rigidity along the lines of Gali (2015), mostly chapter 6, and Walsh (2010), chapter 8. The rest of EZ and RoW are much simplified.

1.1 Income, Prices and Demand

Income: Technology is linear homogeneous and subject to productivity shocks as specified in (47) below. Domestic output is

$$Y = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha}. \tag{1}$$

Firms need labor which is a CES composite of differentiated services offered by specialized workers. A total amount of labor L_t requires services L_{jt} of type j,

$$L_t = \left[\int_0^1 L_{jt}^{(\sigma-1)/\sigma} dj \right]^{\sigma/(\sigma-1)}, \quad \sigma > 1. \tag{2}$$

Firms minimize wage costs, $\min_{L_{jt}} \int_0^1 w_{jt} L_{jt} dj$, subject to technology L_t and wages w_{jt} set by households. By standard steps, demand functions for labor services are

$$L_{jt} = \left(w_t^L / w_{jt} \right)^{\sigma} L_t, \quad w_t^L = \left[\int_0^1 w_{jt}^{1-\sigma} dj \right]^{1/(1-\sigma)}, \quad \int_0^1 w_{jt} L_{jt} dj = w_t^L L_t. \tag{3}$$

Total costs are $w_t^L L_t$ and w_t^L is a nominal wage index.

Goods Demand: Income finances investment and consumption. Consumers buy a composite quantity $\bar{C}_t$ of domestic and foreign goods and must spend $\bar{P}_t \bar{C}_t$. The price index $\bar{P}_t$ is the minimum cost per unit of the composite good. Firms invest and must acquire capital goods $\bar{I}_t$, and spend $\bar{P}_t \bar{I}_t$. We assume that households and firms use the same composite good and thus face the same price index $\bar{P}_t$.

Households of region i consume a basket of final goods C_t^{ij} from different regions. The index j refers the origin country (production). We think of Italy i (home), the rest of the

Eurozone e, and other countries o (RoW). In most cases, we suppress the index i so that $C = C^{ii}$ denotes demand for the home good, and C^{ie} and C^{io} are imports, giving utility

$$\bar{C}_t = \left[\sum_j \left(s^j\right)^{1/\sigma^r} \left(C_t^{ij}\right)^{(\sigma^r-1)/\sigma^r}\right]^{\sigma^r/(\sigma^r-1)}, \quad \sum_j s^j = 1, \quad j \in \{i, e, o\}. \tag{4}$$

Expenditure minimization yields a price index $\bar{P}_t$ and a budget $\sum_j P_t^{ij} C_t^{ij}$ where P_t^{ij} are demand prices paid by domestic consumers for home and foreign produced goods. Consumption thus leads to goods demand

$$C_t^{ij} = s^j \left(\bar{P}_t / P_t^{ij}\right)^{\sigma^r} \bar{C}_t, \quad \bar{P}_t = \left[\sum_j s^j \left(P_t^{ij}\right)^{1-\sigma^r}\right]^{1/(1-\sigma^r)}, \quad \sum_j P_t^{ij} C_t^{ij} = \bar{P}_t \bar{C}_t. \tag{5}$$

Exchange rates relate import prices in domestic currency to the foreign producer prices P_t^e and P^o in foreign currency,

$$P_t^{ie} = e_t^{ie} \cdot P_t^e, \quad P_t^{io} = e_t^{io} \cdot P^o, \quad e_t^{eo} \equiv e_t^{io} / e_t^{ie}. \tag{6}$$

Suppose i (Italy) uses Lire, e uses Euros and o Dollars. The exchange rate converts 1 Euro and 1 Dollar into e_t^{ie} and e_t^{io} Lire. Lira prices for imports are P_t^{ie} and P_t^{io} where foreign producer prices P_t^e and P^o are in foreign currency. The inverse rate converts 1 Lira into $1/e_t^{ie}$ Euros and $1/e_t^{io}$ Dollars. If the Eurozone e imports from Italy, it pays a Euro price P_t/e_t^{ie}, given a producer price in Lire of P_t. RoW pays a Dollar price P_t/e_t^{io} for imports from Italy. By transitivity, the Euro Dollar exchange rate is $e_t^{eo} \equiv e_t^{io}/e_t^{ie}$, i.e., one Dollar buys e_t^{io} Lire, and one Lira gives $1/e_t^{ie}$ Euros, so that one Dollar in the end buys e_t^{eo} Euros. When Italy is part of the Euro Area and shares the same currency with the rest of the Eurozone, it faces a fixed exchange rate $e_t^{ie} = 1$.

1.2 Consumption and Money Demand

The household is an extended family with individuals $j \in [0, 1]$, each offering specialized labor services $L_{jt} = N_{jt}\bar{N}$, where $\bar{N}$ is household size and N_{jt} is labor supply per capita. A household of type j is a monopolist over her specialized labor services. Once labor earnings of each type is optimally determined, the family pools all income and chooses consumption and money holdings.

Preferences for consumption $\bar{C}_t$, labor supply and *real* money balances $\bar{M}_t$ are

$$V_t^h = u\left(\bar{C}_t, \{N_{j,t}\}\bar{N}, \bar{M}_t\right) + \beta_t V_{t+1}^h. \tag{7}$$

Consumption is a composite of home and import goods. Household size is $\bar{N}$, and N_{jt} is labor supply per capita of type j, giving $N_{jt}\bar{N}$ in total. The family pools income. The assumption on preferences eliminates intertemporal substitution in labor supply,

$$u_t = \frac{H_t^{1-\frac{1}{\sigma^c}}}{1-\frac{1}{\sigma^c}} + m_t^{\frac{1}{\sigma^m}}\frac{\bar{M}_t^{1-\frac{1}{\sigma^m}}}{1-\frac{1}{\sigma^m}}, \quad H_t = \bar{C}_t - \Phi\left(\{N_{j,t}\}\right), \quad \Phi_t \equiv \phi_t^{-\frac{1}{\eta}}\frac{\int_0^1 N_{jt}^{1+\frac{1}{\eta}}\bar{N}dj}{1+\frac{1}{\eta}}. \tag{8}$$

Autoregressive shock processes induce fluctuations in labor supply, money demand and the discount rate, $m_t = (1-\rho^m)\bar{m} + \rho^m m_{t-1} + \varepsilon_t^m$, $\phi_t = (1-\rho)\bar{\phi} + \rho\phi_{t-1} + \varepsilon_t^\phi$ as well as $\beta_t = (1-\rho)\bar{\beta} + \rho\beta_{t-1} + \varepsilon_t^\beta$. We record $u_{H,t} = u_{\bar{C},t}$ and

$$u_{\bar{C},t} = H_t^{-\frac{1}{\sigma^c}}, \quad u_{\bar{M},t} = \left(m_t/\bar{M}_t\right)^{\frac{1}{\sigma^m}}, \quad \frac{u_{\bar{M},t}}{u_{\bar{C},t}} = \left(m_t/\bar{M}_t\right)^{\frac{1}{\sigma^m}} H_t^{\frac{1}{\sigma^c}}, \quad -\frac{u_{N_{jt}}}{u_{\bar{C},t}} = \left(N_{jt}/\phi_t\right)^{\frac{1}{\eta}}\bar{N}.$$

Wage Setting: Firms demand specialized labor services of type j. Individual j faces demand L_{jt} for her labor type. Being a monopolist, $N_{jt}\bar{N} = L_{jt}$, she sets a wage to exploit market power. Being one among many close substitutes, she takes the wage index w_t^L and aggregate labor demand L_t as given. The demand schedule in (3), expressed per capita, exhibits a constant wage elasticity,

$$N_{jt} = \left(w_t^L/w_{jt}\right)^\sigma L_t/\bar{N}, \quad \sigma = -\frac{w_{jt}}{N_{jt}}\frac{\partial N_{jt}}{\partial w_{jt}} > 1. \tag{9}$$

For $\sigma \to \infty$, demand is infinitely elastic, leading to standard labor supply choice. Instead of wage setting, individuals would be forced to accept the market wage.

At any date, a *random selection* of workers, a fraction $1-\omega$, can set wages. The remaining fraction ω is stuck with a wage set in the past. In period t, an opportunity for new wage setting arrives with probability $f_{t,t} = 1-\omega$. The household will be stuck with that wage with probability ω^i for another i periods. With probability $f_{t+i,t} = (1-\omega)\omega^i$, she will still have to charge the same wage at date $t+i$. These probabilities must add up to one, $\sum_{i\geq 0} f_{t,t+i} = \sum_{i\geq 0} (1-\omega)\,\omega^i = 1$.

Looking into the past, $f_{t-i,t} = (1-\omega)\,\omega^i$ is the probability that a wage was set in period $t-i$ and is still binding today since it was never changed since then and remained in place over i periods. Again, these probabilities add up to one,

$$f_{t-i,t} = (1-\omega)\,\omega^i, \quad \textstyle\sum_{i\geq 0} f_{t-i,t} = \sum_{i\geq 0} (1-\omega)\,\omega^i = 1. \tag{10}$$

Chances are *identically and independently distributed*. By the law of large numbers, the probability of wage setting is equal to the fraction of workers that can adjust. In a cross-section, employees (or their union) of type j have set wages either today or at some earlier date. In period $t-i$, a fraction $1-\omega$ has set a new wage, and of these, a fraction $f_{t-i,t} = (1-\omega)\,\omega^i$ has adjusted never since then and is stuck today with that same wage. Given symmetry within vintages, a person of type j in period t earns a wage

$$w_{jt} \in \left\{ \begin{array}{ccccc} w_{t,t} = w_t^*, & w_{t-1,t} = w_{t-1}^*, & w_{t-2,t} = w_{t-2}^*, & \ldots & w_{t-i,t} = w_{t-i}^*, \\ 1-\omega & (1-\omega)\,\omega & (1-\omega)\,\omega^2 & & (1-\omega)\,\omega^i. \end{array} \right\}$$

Each vintage $t-i$ includes many agents j. The control w_t^* becomes a state for the next period and all future ones until the next opportunity for wage setting arrives. To find the value of this state, we introduce a state variable w_{t-1}^*, see below. When a person of type j sets an optimal wage in period t, she will earn a wage next period equal to

$$w_{t,t+1} = \left\{ \begin{array}{ll} w_t^* & \text{with prob. } \omega, \\ w_{t+1}^* & \text{with prob. } 1-\omega, \end{array} \right. \qquad w_{t-1,t} = \left\{ \begin{array}{ll} w_{t-1}^* & \text{with prob. } \omega, \\ w_t^* & \text{with prob. } 1-\omega. \end{array} \right. \tag{11}$$

Similarly, a person of vintage $t-1$ continues to earn w_{t-1}^* when she cannot set a new wage (with probability ω), or she earns w_t^* when she is given a chance to revise (with probability $1-\omega$). The control variable w_t^* becomes a state next period with probability ω, and the control w_{t-1}^* is a state this period.

All agents j of the same vintage $t-i$ are identical which gives the following identity. Only the members of the newest vintage can set their optimal wage w_t^*:

$$\Phi_t \equiv \phi_t^{-1/\eta} \cdot \frac{\int_0^1 N_{jt}^{1+1/\eta} \bar{N} dj}{1+1/\eta} = \frac{\bar{N}/\phi_t^{1/\eta}}{1+1/\eta} \cdot \textstyle\sum_{i\geq 0} (1-\omega)\,\omega^i \cdot N_{t-i,t}^{1+1/\eta} dj. \tag{12}$$

Budget Constraint: The family smooths income risk and cares about total earnings. Households pay wage and consumption taxes at rates τ_t and τ_t^c, and are able to reduce tax liability by T_t^l due to tax avoidance. They receive social transfers E_t and seignorage T_t^M and collect capital income χ_t^A (dividends, interest on government bonds etc., as listed in 1.7 below) from all sources other than residual savings A_t. The nominal budget is

$$\begin{aligned} A_t &= (1+i_{t-1}) A_{t-1} + \int_0^1 (1-\tau_t) w_{jt} N_{jt} \bar{N} dj + E_t + T_t^l \\ &: +\chi_t^A + T_t^M - (M_t - M_{t-1}) - (1+\tau_t^c) \bar{P}_t \bar{C}_t. \end{aligned} \quad (13)$$

The dating convention is that stocks M_t and A_t are measured at the end of period so that M_{t-1} and A_{t-1} are beginning of t, or end of $t-1$. Nominal money holdings M_{t-1} and real money balances are related by $M_{t-1} \equiv \bar{M}_{t-1}\bar{P}_{t-1}$ and $M_t \equiv \bar{M}_t \bar{P}_t$.

Optimization: Current utility depends on real money balances $\bar{M}_t = M_t/\bar{P}_t$. Noting the value function $V_{t+1}^h = V^h\left(A_t, M_t, w_t^* \bar{N}\right)$, the Bellman problem is

$$V_t^h = V^h\left(A_{t-1}, M_{t-1}, w_{t-1}^* \bar{N}\right) = \max_{\bar{C}_t, M_t, w_t^*} u\left(\bar{C}_t, \{N_{j,t}\}\bar{N}, \bar{M}_t\right) + \beta_t V_{t+1}^h,$$

subject to (9), (11), (13). The control w_t^* refers only to types j in vintage t, $N_{jt} \in N_{t,t}$ which receive a new opportunity for wage setting. Define shadow prices $\lambda_{t+1} \equiv dV_{t+1}/dA_t$, $\lambda_t^M \equiv dV_t/dM_{t-1}$ and $dV_t/dw_{t-1}^* = \mu_t^*$. Optimality conditions for $\bar{C}_t, M_t, w_t^*$ are

$$\begin{aligned} \bar{C}_t &: u_{\bar{C},t} = \beta_t \lambda_{t+1} (1+\tau_t^c) \bar{P}_t, \\ M_t &: u_{\bar{M},t} = \beta_t \left(\lambda_{t+1} - \lambda_{t+1}^M\right) \bar{P}_t, \\ w_t^* &: 0 = u_{N_{t,t}} \bar{N} \frac{\partial N_{t,t}}{\partial w_t^*} + \beta_t \lambda_{t+1} (1-\tau_t) \bar{N} \frac{\partial (w_t^* N_{t,t})}{\partial w_t^*} + \omega \cdot \beta_t \mu_{t+1}^*, \end{aligned} \quad (14)$$

and envelope conditions for $A_{t-1}, M_{t-1}, w_{t-1}^*$ are (see section 1.3 for w_t^* and w_{t-1}^*)

$$\begin{aligned} A_{t-1} &: \lambda_t = \beta_t \lambda_{t+1} (1+i_{t-1}), \\ M_{t-1} &: \lambda_t^M = \beta_t \lambda_{t+1}, \\ w_{t-1}^* &: \mu_t^* = u_{N_{t-1,t}} \bar{N} \frac{\partial N_{t-1,t}}{\partial w_{t-1}^*} + \beta_t \lambda_{t+1} (1-\tau_t) \bar{N} \frac{\partial \left(w_{t-1}^* N_{t-1,t}\right)}{\partial w_{t-1}^*} + \omega \cdot \beta_t \mu_{t+1}^*. \end{aligned}$$

Shift forward (14.iv) by one period, multiply by $\beta_t (1+\tau_t^c) \bar{P}_t$, rearrange terms to use (i) on both sides and get the Euler condition

$$u_{\bar{C},t} = \beta_t (1+r_t) \cdot u_{\bar{C},t+1}, \quad 1 = (\beta_t (1+r_t))^{\sigma^c} \cdot H_t / H_{t+1}. \tag{15}$$

The Fisher equation relates interest and inflation rates by $r_t \approx i_t - \pi_t$ or, more precisely,

$$1 + r_t = \frac{1+i_t}{1+\pi_t}, \quad 1 + \pi_t = \frac{\left(1+\tau_{t+1}^c\right) \bar{P}_{t+1}}{(1+\tau_t^c) \bar{P}_t}. \tag{16}$$

Saving 1 Euro reduces marginal utility today by $u_{\bar{C},t} / \left[(1+\tau_t^c) \bar{P}_t\right]$. One Euro gives $1+i_t$ tomorrow, raising marginal utility by $(1+i_t) u_{\bar{C},t+1} / \left[\left(1+\tau_{t+1}^c\right) \bar{P}_{t+1}\right]$.[1] Discounting back and comparing with today's utility loss gives the intertemporal allocation of H_t. A higher real interest tilts full consumption to the future and boosts savings today.

By (iv-v), $\lambda_t^M = \lambda_t / (1+i_{t-1})$. Using this (shifted to t) together with the Fisher equation in (ii) and combining with (i) gives the tangency condition for money demand,

$$\frac{u_{\bar{M},t}}{u_{\bar{C},t}} = \frac{i_t}{(1+\tau_t^c)(1+i_t)}, \quad \bar{M}_t = m_t \cdot \left[\frac{1+i_t}{i_t} (1+\tau_t^c) H_t^{1/\sigma^c}\right]^{\sigma^m}. \tag{17}$$

Holding 1 Euro more money today raises marginal utility by $u_{\bar{M},t} / \bar{P}_t$ but reduces current consumption and marginal utility by $u_{\bar{C},t} / \left[(1+\tau_t^c) \bar{P}_t\right]$. A higher consumption tax thus leads households to shift from full consumption to money holdings. In addition, money demand comes with an opportunity cost equal to the return that could have been obtained if it were invested in the market at a rate i_t, or $i_t / (1+i_t)$ in present value. Balancing marginal gains and costs gives (17).

Return Structure: Households own firms and banks and hold deposits and sovereign debt and receive capital income χ_t^A. Interest income on last period savings is i_{t-1}^d and i_{t-1}^g on deposit and sovereign bond holdings. Dividends χ^k and χ^b reflect the equity returns i_{t-1}^k and i_{t-1}^b that households required in $t-1$. In period t, current savings are invested in (residual) bonds, which yield income $1+i_t$ next period. Similarly, i_t^k, i_t^b, i_t^g and i_t^d are

[1] Note that i_{t-1} is interest accruing from beginning to end of period t, flowing from 1 Euro saved in $t-1$. In contrast, $1+i_t$ generates income next period on 1 Euro saved today.

the required returns at the end of t, giving income $1 + i_t^d$ next period etc. Investors are indifferent to investing in these alternative assets if they are compensated for risk and other characteristics with appropriate markups,

$$\begin{aligned} i_t^k &= i_t + \theta_t^k, \quad \theta_t^k = \left(1 - \rho^\theta\right) \bar{\theta}^k + \rho^\theta \theta_{t-1}^k + \varepsilon_t^k, \\ i_t^b &= i_t + \theta_t^b, \quad \theta_t^b = \left(1 - \rho^\theta\right) \bar{\theta}^b + \rho^\theta \theta_{t-1}^b + \varepsilon_t^b, \\ i_t^g &= i_t + \theta_t^g, \quad \theta_t^g = \left(1 - \rho^\theta\right) \bar{\theta}^g + \rho^\theta \theta_{t-1}^g + \varepsilon_t^g, \\ i_t^d &= i_t + \theta_t^d, \quad \theta_t^d = 1 - \rho^\theta + \rho^\theta \theta_{t-1}^d + \varepsilon_t^d. \end{aligned} \tag{18}$$

Assets are perfect substitutes up to compensating return differentials which may be subject to shocks reflecting increasing perceived risk, loss of investor confidence etc. A natural ordering is $i_t^k, i_t^b > i_t^l > i_t^g \geq i_t^d \geq i_t$.

1.3 Wage Setting

Labor supply follows from optimal wage setting in (14.iii). When receiving a new opportunity for wage adjustment, an agent can exploit local market power for her specialized labor services, with three effects on optimal choice. By the first term, insisting on a higher wage reduces demand for labor services and thereby reduces marginal effort cost of labor supply. Second, a higher wage reduces disposable wage earnings and thereby end of period financial wealth which raises future consumption with a discounted value $\beta_t \lambda_{t+1}$ per Euro. Third, by (11), setting a higher wage $w_{t,t} = w_t^*$ today also raises tomorrow's wage of that same agent by $w_{t,t+1} = w_t^*$ with probability ω (with probability $1 - \omega$, she is able to set a new wage), with a discounted value $\beta_t \mu_t^*$ per unit. A similar interpretation applies to the envelope condition (14.vi).

To derive the labor supply condition, we use the demand elasticity σ and substitute for $\frac{\partial N_{t,t}}{\partial w_t^*} = -\frac{N_{t,t}}{w_t^*}\sigma$ and $\frac{\partial (w_t^* N_{t,t})}{\partial w_t^*} = N_{t,t} + w_t^* \frac{\partial N_{t,t}}{\partial w_t^*} = -(\sigma - 1) N_{t,t}$ in (14.iii). Multiply by

$\frac{w_t^*}{N_{t,t}}$, use (i), divide by $(\sigma-1)\bar{N}u_{\bar{C},t}$ and substitute $MRS_{t,t} \equiv -\frac{u_{N_{t,t}}}{u_{\bar{C},t}}$,

$$\begin{aligned}\frac{(1-\tau_t)\,w_t^*}{(1+\tau_t^c)\,\bar{P}_t} &= \frac{\sigma}{\sigma-1}MRS_{t,t} + \frac{w_t^*}{N_{t,t}}\omega\frac{\beta_t u_{\bar{C},t+1}}{u_{\bar{C},t}}\frac{\mu_{t+1}^*/\bar{N}}{(\sigma-1)\,u_{\bar{C},t+1}},\\ \frac{\mu_t^*/\bar{N}}{(\sigma-1)\,u_{\bar{C},t}} &= \left[\frac{\sigma}{\sigma-1}MRS_{t-1,t} - \frac{(1-\tau_t)\,w_{t-1}^*}{(1+\tau_t^c)\,\bar{P}_t}\right]\frac{N_{t-1,t}}{w_{t-1}^*} + \omega\frac{\beta_t u_{\bar{C},t+1}}{u_{\bar{C},t}}\frac{\mu_{t+1}^*/\bar{N}}{(\sigma-1)\,u_{\bar{C},t+1}}.\end{aligned}$$

Doing similar steps gives the second equation from the envelope condition (14.vi). The Euler equation is $\frac{\beta_t u_{\bar{C},t+1}}{u_{\bar{C},t}} = \frac{1}{1+r_t}$. Define $\mu_{t+1}^w \equiv \frac{\mu_{t+1}^*/\bar{N}}{(\sigma-1)u_{\bar{C},t+1}}$ and get the final solution

$$\begin{aligned}\frac{(1-\tau_t)\,w_t^*}{(1+\tau_t^c)\,\bar{P}_t} &= \frac{\sigma}{\sigma-1}\cdot MRS_{t,t} + \frac{w_t^*}{N_{t,t}}\omega\frac{\mu_{t+1}^w}{1+r_t},\\ \mu_t^w &= \left[\frac{\sigma}{\sigma-1}\cdot MRS_{t-1,t} - \frac{(1-\tau_t)\,w_{t-1}^*}{(1+\tau_t^c)\,\bar{P}_t}\right]\frac{N_{t-1,t}}{w_{t-1}^*} + \omega\frac{\mu_{t+1}^w}{1+r_t}.\end{aligned} \tag{19}$$

Computations require $N_{t,t} = \left(w_t^L/w_t^*\right)^\sigma L_t/\bar{N}$ and $N_{t-1,t} = \left(w_t^L/w_{t-1}^*\right)^\sigma L_t/\bar{N}$ as well as

$$MRS_{t,t} = -\frac{u_{N_{t,t}}}{u_{\bar{C},t}} = \left(\frac{N_{t,t}}{\phi_t}\right)^{\frac{1}{\eta}}\bar{N}, \quad MRS_{t-1,t} = \left(\frac{N_{t-1,t}}{\phi_t}\right)^{\frac{1}{\eta}}\bar{N} = \left(\frac{w_t^*}{w_{t-1}^*}\right)^{\frac{\sigma}{\eta}}\cdot MRS_{t,t}. \tag{i}$$

Define $X_{t,t+i} \equiv \left[\frac{\sigma}{\sigma-1}MRS_{t,t+i} - \frac{(1-\tau_{t+i})w_t^*}{\left(1+\tau_{t+1}^c\right)\bar{P}_{t+i}}\right]\frac{N_{t,t+i}}{w_t^*}$ where the currently set wage w_t^* is the same for all future periods $t+i$. Shift forward (19.ii) one period, write the system as $0 = X_{t,t} + \frac{\omega}{1+r_t}\mu_{t+1}^w$ and $\mu_{t+1}^w = X_{t,t+1} + \frac{\omega}{1+r_{t+1}}\mu_{t+2}^w$, and solve forward to obtain $0 = \sum_{i\geq 0} X_{t,t+i}\Lambda_{t,t+i}$, see also the Appendix. The discount factor is $\Lambda_{t,t+i} = \prod_{j=0}^{i-1}\frac{\omega}{1+r_{t+j}}$ with $\Lambda_{t,t} = 1$. Use $\sum_{i\geq 1} X_{t,t+i}\Lambda_{t,t+i} = -X_{t,t}$ and multiply both sides by $\frac{\sigma-1}{\sigma}\frac{w_t^*}{N_{t,t}}$, giving

$$\frac{(1-\tau_t)\,w_t^*}{(1+\tau_t^c)\,\bar{P}_t} = \frac{\sigma}{\sigma-1}MRS_{t,t} + \sum_{i\geq 1}\left[\frac{\sigma}{\sigma-1}MRS_{t,t+i} - \frac{(1-\tau_{t+i})\,w_t^*}{\left(1+\tau_{t+1}^c\right)\bar{P}_{t+i}}\right]\frac{N_{t,t+i}}{N_{t,t}}\cdot\Lambda_{t,t+i}. \tag{ii}$$

The marginal rate of substitution measures the opportunity cost of work which must be compensated by a sufficiently high real wage. Given wage rigidity, households set a wage today that *on average* just compensates for the opportunity cost of work, plus a markup to exploit monopoly power. The average is taken over the expected period of wage duration (the wage stays fixed until the next wage setting opportunity arrives), weighted by relative labor supply $\frac{N_{t,t+i}}{N_{t,t}}$. Setting a constant wage w_t^* thus leads to positive rents $\frac{\sigma-1}{\sigma}\frac{(1-\tau_{t+i})w_t^*}{\left(1+\tau_{t+1}^c\right)\bar{P}_{t+i}} - MRS_{t,t+i}$ in some periods and negative ones in others, with the weighted average being zero. In a SS, labor supply and rents are constant over time, so that wage setting implies zero rents in all periods, see (20) below. In general, the optimality condition (19.i) gives $MRS_{t,t+i} = \left(N_{t,t+i}/\phi_{t+i}\right)^{1/\eta}$. Holding $\phi_{t+i} = \phi$ constant implies $N_{t,t+i} \gtreqless N_{t,t} \Leftrightarrow MRS_{t,t+i} \gtreqless MRS_{t,t}$. When labor gets relatively more scarce and requires relatively more work in the future, households set a higher wage today to compensate at least partly for higher future opportunity cost. On net, when labor gets scarce in the future, future rents are positive and current rents negative, with optimal wage setting reducing the deviation from the average.

Stationary Solution: In a SS, wages are constant, giving $w_t^* = w_{t-1}^* = w_{t-i}^*$, which also implies that $MRS_{t-i,t}$ and $N_{t-i,t}$ are the same for all vintages. Intuitively, wage rigidity disappears in a SS. Substituting (19.i) into (ii) implies a static mark-up

$$\frac{(1-\tau)\, w^*}{(1+\tau^c)\, \bar{P}} = \frac{\sigma}{\sigma-1} \cdot MRS, \quad \mu^w = 0. \tag{20}$$

Similarly, if wages were flexible in all periods ($\omega = 0$), they would all be optimally set at $w_{t-i,t} = w_t^*$. Given $\omega = 0$, (19.i) gives a static wage mark-up, $\frac{(1-\tau_t)w_t^*}{(1+\tau_t^c)\bar{P}_t} = \frac{\sigma}{\sigma-1} \cdot MRS_{t,t}$, so that $\mu_{t+1}^w = \mu_t^w = 0$.[2] Although derived rather differently as part of the Bellman problem, the solution exactly corresponds to Gali (2015), see the Appendix for details.

Aggregation: Wages and employment are heterogeneous. They differ by their vintage but are symmetric within each cohort. To get the effect of new and old wages on the wage index $w_t^L = \left[\int_0^1 w_{j,t}^{1-\sigma} dj\right]^{1/(1-\sigma)}$, we use the distribution of types j in (11) and arrange wages according to vintages, $\left(w_t^L\right)^{1-\sigma} = \sum_{i \geq 0} (1-\omega)\, \omega^i w_{t-i,t}^{1-\sigma}$, giving

$$\left(w_t^L\right)^{1-\sigma} = (1-\omega)\, w_{t,t}^{1-\sigma} + \omega \cdot (1-\omega) \left[w_{t-1,t-1}^{1-\sigma} + \omega w_{t-2,t-1}^{1-\sigma} + \omega^2 w_{t-3,t-1}^{1-\sigma} + ...\right].$$

A wage set at date $t-i$ is constant thereafter until the next wage setting. The second line thus uses $w_{t-i,t} = w_{t-i,t-1}$, and the definition of the wage index in $t-1$, which gives

$$\left(w_t^L\right)^{1-\sigma} = (1-\omega) \cdot w_{t,t}^{1-\sigma} + \omega \cdot \left(w_{t-1}^L\right)^{1-\sigma}. \tag{21}$$

[2]Intertemporal substitution in labor supply is shut off since $MRS_{j,t} \equiv -\frac{u_{N_{j,t}}}{u_{\bar{C},t}} = \left(N_{jt}/\phi_t\right)^{1/\eta}$ is independent of forward looking consumption. Using this, we obtain a symmetric static labor supply schedule $N_{jt} = \phi_t \left[\frac{\sigma^l-1}{\sigma^l} \frac{(1-\tau_t)w_t^*}{(1+\tau_t^c)\bar{P}_t}\right]^\eta$ which exclusively depends on the current real wage.

The state w_{t-1}^L and current wages $w_{t,t}$ give the evolution of the wage index w_t^L. Total wage earnings are then $w_t^L L_t = \int_0^1 w_{jt} N_{jt} \bar{N} dj$.

The disutility of work as defined in (12) is $\Phi_t = \frac{\bar{N}\phi_t^{-1/\eta}}{1+1/\eta} \cdot \int_0^1 N_{jt}^{1+1/\eta} dj$. Substituting demand $N_{jt} = \left(w_t^L / w_{jt}\right)^\sigma L_t / \bar{N}$ in (9) yields

$$\Phi_t = \frac{\left(w_t^L\right)^{(1+1/\eta)\sigma}}{1+1/\eta} \frac{L_t^{1+1/\eta}}{\left(\phi_t \bar{N}\right)^{1/\eta}} \cdot \sum_{i \geq 0} (1-\omega)\, \omega^i \left(w_{t-i,t}\right)^{-(1+1/\eta)\sigma}.$$

Defining $\bar{w}_t^L$ in parallel to (21) gives $\Phi_t = \frac{\bar{N}/\phi_t^{1/\eta}}{1+1/\eta} \left[\left(w_t^L / \bar{w}_t^L\right)^\sigma L_t / \bar{N}\right]^{1+1/\eta}$ and

$$\left(\bar{w}_t^L\right)^{-(1+1/\eta)\sigma} = (1-\omega)\left(w_{t,t}\right)^{-(1+1/\eta)\sigma} + \omega \left(\bar{w}_{t-1}^L\right)^{-(1+1/\eta)\sigma}. \tag{22}$$

Numerical Procedure: Computing (19) needs $MRS_{j,t} \equiv -\frac{u_{N_{j,t}}}{u_{\bar{C},t}} = \left(N_{jt}/\phi_t\right)^{1/\eta}$ for vintages $j = t$ and $j = t-1$. Employment of a wage setter is $N_{t,t} = \left(w_t^L / w_t^*\right)^\sigma L_t / \bar{N}$, resulting in $MRS_{t,t}$. Noting $w_{t-1,t} = w_{t-1}^*$ similarly gives $N_{t-1,t} = \left(w_t^L / w_{t-1}^*\right)^\sigma L_t / \bar{N}$ and $MRS_{t-1,t}$. The expected variable μ_{t+1}^w thus gives w_t^* from (19.i). Given the predetermined variable w_{t-1}^*, we compute an update μ_t^w for the expected variable. Finally, composite goods demand is $\bar{C}_t = H_t + \Phi_t$ and follows from the intertemporal allocation H_t and aggregate effort cost of work Φ_t computed prior to (22).

1.4 Investment and Debt Financing

Financial Identities: Technology is stated in (1). Firms also incur installation costs J_t, i.e., they need $\bar{I}_t + J_t$ composite goods to install $\bar{I}_t$ units of new equipment,

$$J_t = \frac{\psi^k}{2} K_{t-1} \left(\bar{I}_t / K_{t-1} - \delta^k - s_t\right)^2. \tag{23}$$

Define $J_{K,t} \equiv \frac{dJ_t}{dK_{t-1}}$ and $J_{\bar{I},t}$, so $J_{K,t} = -\frac{1}{2}\psi^k \left(\bar{I}_t / K_{t-1} - \delta^k - s_t\right)\left(\bar{I}_t / K_{t-1} + \delta^k + s_t\right)$ and $J_{\bar{I},t} = \psi^k \left(\bar{I}_t / K_{t-1} - \delta^k - s_t\right)$. Note $\bar{I}_t J_{I,t} + K_{t-1} J_{K,t} = J_t$ by linear homogeneity. Firms thus spend $\left(J_t + \bar{I}_t\right) \bar{P}_t$ on new capital goods.

Capital depreciates at a rate $\delta^k + s_t$ where s_t represents losses due to bankruptcy. Capital and debt accumulate by

$$K_t = \bar{I}_t + \left(1 - \delta^k - s_t\right) K_{t-1}, \quad B_t^l = N_t^l + \left(1 - \delta^k - s_t\right) B_{t-1}^l. \tag{24}$$

Firms inherit assets and liabilities, $\bar{P}_{t-1}K_{t-1} = B^l_{t-1} + E^k_{t-1}$, equal to debt B^l_{t-1} and equity E^k_{t-1}. We assume that firms repay debt with normal depreciation. Depreciation and liquidation of bad loans diminish total assets by $\left(\delta^k + s_t\right) \bar{P}_{t-1}K_{t-1}$. In parallel, outstanding debt shrinks by normal repayment and by debt relief in the bankruptcy process, in total by $\left(\delta^k + s_t\right) B^l_{t-1}$.[3] With assets and debt shrinking by the same factor, equity also declines by $\left(\delta^k + s_t\right) E^b_{t-1}$. However, due to changing capital goods prices $\bar{P}_t$, equity value changes due to capital gains on surviving assets, $\left(\bar{P}_t - \bar{P}_{t-1}\right)\left(1 - \delta^k - s_t\right) K_{t-1}$. All in all, equity accumulates together with retained earnings N^k_t by

$$E^k_t = N^k_t + \left(\bar{P}_t - \bar{P}_{t-1}\right)\left(1 - \delta^k - s_t\right) K_{t-1} + \left(1 - \delta^k - s_t\right) E^b_{t-1}. \tag{25}$$

Using (24-25), the balance sheet changes by $\nabla\left(\bar{P}_t K_t\right) = \nabla B^l_t + \nabla E^k_t$, giving

$$\bar{P}_t K_t = B^l_t + E^k_t \quad \Rightarrow \quad \bar{P}_t \bar{I}_t = N^l_t + N^k_t. \tag{26}$$

New loans N^l_t and retained earnings N^k_t finance spending $\bar{P}_t \bar{I}_t$ on new equipment.

The loan rate i^l_t reflects current credit market equilibrium. Installation costs $J_t \bar{P}_t$ are financed out of cash-flow, defined as $\pi^k_t \equiv \left(1 - \tau^k_t\right)\left(P_t Y_t - J_t \bar{P}_t - w^L_t L_t - i^l_t B^l_{t-1}\right)$. The fundamental financial identity $\pi^k_t + N^l_t = \chi^k_t + \bar{P}_t \bar{I}_t + \delta^k B^l_{t-1}$ relates inflows and outflows, leaving dividends of

$$\begin{aligned}
\chi^k_t &= P_t Y_t - J_t \bar{P}_t - w^L_t L_t - i^l_t B^l_{t-1} - \tau^k_t T^k_t - \bar{P}_t \bar{I}_t + N^l_t - \delta^k B^l_{t-1}, \\
T^k_t &= P_t Y_t - J_t \bar{P}_t - w^L_t L_t - i^l_t B^l_{t-1}, \\
\chi^k_t &= \left(1 - \tau^k_t\right)\left(P_t Y_t - J_t \bar{P}_t - w^L_t L_t - i^l_t B^l_{t-1}\right) - \bar{P}_t \bar{I}_t + N^l_t - \delta^k B^l_{t-1}.
\end{aligned} \tag{27}$$

We lump together corporate and personal taxes on capital income which gets taxed with a tax rate τ^k_t. The tax base T^k_t allows for deduction of interest expenses on debt. Cash-flow is used for dividends, debt repayment and retained earnings, $\pi^k_t = \chi^k_t + \delta^k B^l_{t-1} + N^k_t$.

[3] The business sector consists of many independent firms that operate as separate legal entities protected by limited liability. When $s_t \bar{P}_{t-1} K_{t-1}$ assets are liquidated, banks must write off credit $s_t B^l_{t-1}$. They seize all assets of the firm and keep liquidation revenues to reduce effective losses, see below. Failed firms, however, lose the entire equity capital.

The debt capacity of firms is limited. Bank credit is in exchange for collateral, equal to a fraction of total assets. We thus postulate a leverage constraint

$$B_t^l \leq b^k \cdot \bar{P}_t K_t. \tag{28}$$

The required return on equity exceeds the deposit rate, $i_t^k > i_t$. Given net dividends, firm value V_t^k (in nominal terms) is

$$V_t^k = \chi_t^k + V_{t+1}^k / \left(1 + i_t^k\right). \tag{29}$$

OPTIMIZATION: Value function is $V_{t+1}^k = V\left(K_t, B_t^l\right)$, the Bellman problem

$$\begin{aligned} V\left(K_t, B_t^l\right) &= \max\nolimits_{\bar{I}_t, N_t^l, L_t} \ \chi_t^k + \left[V_{t+1}^k + \mu_t^k \cdot \left(b^k \bar{P}_t K_t - B_t^l\right)\right] / \left(1 + i_t^k\right) \\ s.t. \ \ \chi_t^k &= \left(1 - \tau_t^k\right)\left[P_t Y_t - J\left(\bar{I}_t, K_{t-1}\right)\bar{P}_t - w_t^L L_t - i_t^l B_{t-1}^l\right] - \bar{P}_t \bar{I}_t + N_t^l - \delta^k B_{t-1}^l, \\ K_t &= \bar{I}_t + \left(1 - \delta^k - s_t\right) K_{t-1}, \quad B_t^l = N_t^l + \left(1 - \delta^k - s_t\right) B_{t-1}^l. \end{aligned} \tag{30}$$

Define shadow prices $\partial V_t^k / \partial K_{t-1} \equiv \lambda_t^k$ and $\partial V_t^k / \partial B_{t-1}^l \equiv -\lambda_t^l$. Optimality conditions are

$$\begin{aligned} L_t &: \ 0 = \left(1 - \tau_t^k\right)\left(P_t F_{L,t} - w_t^L\right), \\ \bar{I}_t &: \ 0 = -\left(1 + \left(1 - \tau_t^k\right) J_{\bar{I},t}\right)\bar{P}_t + \left(\lambda_{t+1}^k + \mu_t^k b^k \bar{P}_t\right) / \left(1 + i_t^k\right), \\ N_t^l &: \ 0 = 1 - \left(\lambda_{t+1}^l + \mu_t^k\right) / \left(1 + i_t^k\right). \end{aligned} \tag{31}$$

Envelope conditions are

$$\begin{aligned} K_{t-1} &: \ \lambda_t^k = \left(1 - \tau_t^k\right)\left(P_t F_{K,t} - J_{K,t}\bar{P}_t\right) + \left(1 - \delta^k - s_t\right)\left(\lambda_{t+1}^k + \mu_t^k b^k \bar{P}_t\right) / \left(1 + i_t^k\right), \\ B_{t-1}^l &: \ \lambda_t^l = \left(1 - \tau_t^k\right) i_t^l + \delta^k + \left(1 - \delta^k - s_t\right)\left(\lambda_{t+1}^l + \mu_t^k\right) / \left(1 + i_t^k\right). \end{aligned} \tag{32}$$

CAPITAL STRUCTURE: Conditions (31.iii) and (32.ii) give λ_t^l. Substituting this for λ_{t+1}^l in (31.iii) gives the multiplier μ_t^k,

$$\lambda_t^l = \left(1 - \tau_t^k\right) i_t^l + 1 - s_t, \quad \mu_t^k = 1 + i_t^k - \lambda_{t+1}^l = i_t^k - \left(1 - \tau_{t+1}^k\right) i_{t+1}^l + s_{t+1} > 0. \tag{33}$$

Given interest rates $i^k > i^l$, the shadow price is positive. The debt constraint binds,

$$\mu_t^k > 0 \quad \Rightarrow \quad B_t^l = b^k \bar{P}_t K_t. \tag{34}$$

By the balance sheet, equity capital is $E_t^k = \left(1 - b^k\right) \bar{P}_t K_t$.

Investment: Multiply (31.iii) with $b^k \bar{P}_t$, subtract from (ii) and divided the result by $\bar{P}_t$. Reflecting Tobin's Q-theory, we have $1 - b^k + \left(1 - \tau_t^k\right) J_{\bar{I},t} = Q_t^k$, or

$$\bar{I}_t = \left[\delta^k + s_t + \frac{Q_t^k - \left(1 - b^k\right)}{\left(1 - \tau_t^k\right) \psi^k}\right] \cdot K_{t-1}, \quad Q_t^k \equiv \frac{\lambda_{t+1}^f}{\left(1 + i_t^k\right) \bar{P}_t}, \tag{35}$$

where $\lambda_{t+1}^f \equiv \lambda_{t+1}^k - b^k \bar{P}_t \lambda_{t+1}^l$. When the marginal value of equity capital, as measured by Q_t^k, exceeds marginal retained earnings $1 - b^k$, net investment becomes positive and augments the capital stock.

To obtain the current value of λ_t^f, multiply (32.ii) by $b^k \bar{P}_t$ and subtract it from (i). Expand the l.h.s. by $+\left(b^k - b^k\right) \bar{P}_{t-1} \lambda_t^l = 0$, substitute λ_t^l from (33) in the term $\left(\bar{P}_t - \bar{P}_{t-1}\right) b^k \lambda_t^l$, use the definition of λ_t^f, and finally get

$$\begin{aligned} \lambda_t^f &= \left(1 - \tau_t^k\right) \left(P_t F_{K,t} - J_{K,t} \bar{P}_t - i_t^l b^k \bar{P}_{t-1}\right) \\ &: \quad - \left(1 - s_t\right) b^k \bar{P}_{t-1} + \left(1 - \delta^k - s_t\right) \left[b^k \bar{P}_t + \lambda_{t+1}^f / \left(1 + i_t^k\right)\right]. \end{aligned} \tag{36}$$

In a SS, $J_{\bar{I}} = J_K = 0$ and $Q^k = 1 - b^k$, dictating $\lambda^f = \left(1 + i^k\right) \left(1 - b^k\right) \bar{P}$. Using this gives $\left(1 + i^k\right) \left(1 - b^k\right) = \left(1 - \tau^k\right) \left(P F_K / \bar{P} - i^l b^k\right) - \left(1 - s\right) b^k + \left(1 - \delta^k - s\right)$, which eventually results in the standard user cost formula,

$$P F_K / \bar{P} = \left(\bar{\imath}^k + \delta^k + s\right) / \left(1 - \tau^k\right), \quad \bar{\imath}^k \equiv i^k \cdot \left(1 - b^k\right) + \left(\left(1 - \tau^k\right) i^l - s\right) \cdot b^k. \tag{37}$$

The real return on capital is equal to weighted funding costs. While the cost of debt is tax deductible, the cost of equity is not and therefore gets inflated by the factor $1 / \left(1 - \tau^k\right)$.

VALUE FUNCTION: The value function satisfies

$$V_t^k = \lambda_t^k K_{t-1} - \lambda_t^l B_{t-1}^l = \lambda_t^f K_{t-1}. \tag{38}$$

To prove this, multiply envelope conditions by stocks, substitute the laws of motion for $\left(1 - \delta^k - s_t\right) K_{t-1} = K_t - \bar{I}_t$ and $\left(1 - \delta^k - s_t\right) B_{t-1}^l = B_t^l - N_t^l$ and apply the optimality conditions to the terms $\bar{I}_t$ and N_t^l. Finally note $\bar{I}_t J_{I,t} + K_{t-1} J_{K,t} = J_t$ and $K_{t-1} F_{K,t} =$

$Y_t - F_{L,t}L_t$ by linear homogeneity and apply $P_t F_{L,t} = w_t^L$:

$$
\begin{aligned}
\lambda_t^k K_{t-1} &= \left(1-\tau_t^k\right)\left(P_t Y_t - w_t^L L_t - J_t \bar{P}_t\right) - \bar{P}_t \bar{I}_t + \mu_t^k \frac{b^k \bar{P}_t K_t}{1+i_t^k} + \frac{\lambda_{t+1}^k K_t}{1+i_t^k}, \\
\lambda_t^l B_{t-1}^l &= \left(1-\tau_t^k\right) i_t^l B_{t-1}^l + \delta^k B_{t-1}^l - N_t^l + \mu_t^k \frac{B_t^l}{1+i_t^k} + \frac{\lambda_{t+1}^l B_t^l}{1+i_t^k}.
\end{aligned}
$$

Subtracting the two equations and noting dividends in (27) gives

$$
\lambda_t^k K_{t-1} - \lambda_t^l B_{t-1}^l = \chi_t^k + \mu_t^k \cdot \frac{b^k \bar{P}_t K_t - B_t^l}{1+i_t^k} + \frac{\lambda_{t+1}^k K_t - \lambda_{t+1}^l B_t^l}{1+i_t^k}. \tag{39}
$$

By the capital constraint, the second term is zero. Comparing the forward solutions of this and of (29) gives the first equality in (38). Noting capital structure $B_{t-1}^l = b^k \bar{P}_{t-1} K_{t-1}$ and the shadow price $\lambda_t^f = \lambda_t^k - b^k \bar{P}_{t-1} \lambda_t^l$ establishes the second equality.

1.5 Fiscal Policy

The government issues long-term bonds and pays a fixed coupon rate $\bar{\imath}$. At the end of t, agents invest in bonds and insist on a return i_t^g, accruing in period $t+1$. The bond price must adjust to deliver this return. Bonds are repaid with probability μ at face value of 1 at the end of $t+1$, and continue with probability $1-\mu$. The price $Q_{j,t}$ is $\left(1+i_t^g\right) Q_{j,t} = \bar{\imath} + \mu + (1-\mu) Q_{j,t+1}$, referring to a vintage j bond issued at date $j \leq t$. Bond prices are symmetric, $Q_{j,t} = Q_t$. New and old bonds have the same repayment probability μ, coupon rate $\bar{\imath}$ and expected return i_t^g. Investing Q_t in a one period bond gives value $\left(1+i_t^g\right) Q_t$ at the end of $t+1$. The bond yields a coupon rate $\bar{\imath}$ next period, repayment of 1 with probability μ at the end of $t+1$, and value Q_{t+1} if not repaid with probability $1-\mu$.[4] Expected duration is $1/\mu$. Solving forward gives $Q_t = \frac{\bar{\imath}+\mu}{1+i_t^g}\left(1 + \frac{1-\mu}{1+i_{t+1}^g} + \frac{1-\mu}{1+i_{t+1}^g}\frac{1-\mu}{1+i_{t+2}^g} + \ldots\right)$. In recursive form,

$$
\left(1+i_t^g\right) Q_t = \bar{\imath} + \mu + (1-\mu) Q_{t+1}, \quad Q = \frac{\bar{\imath}+\mu}{i^g+\mu}. \tag{40}
$$

[4] Alternatively, write $i_t^g Q_t = \bar{\imath} + \mu\left(1-Q_t\right) + (1-\mu)\left(Q_{t+1}-Q_t\right)$. The return next period on holding the bond consists of a dividend $\bar{\imath}$ plus expected capital gains: with probability μ, the bond is repaid at face value, giving a capital gain $1-Q_t$; with probability $1-\mu$, the bond is not repaid and is valued Q_{t+1} at the end of $t+1$, giving a capital gain $Q_{t+1}-Q_t$.

If the coupon rate is equal to the market rate i^g, the bond price is $Q = 1$.

The quantity at the end of t is $B_t^G = B_{t,t}^G + B_{t-1,t}^G + B_{t-2,t}^G + \ldots$. Of each vintage, a fraction μ is repaid which leaves a fraction $1-\mu$ one period later, $B_{j,t}^G = (1-\mu) B_{j,t-1}^G$. Debt $B_t^G = B_{t,t}^G + (1-\mu)\left[B_{t-1,t-1}^G + B_{t-2,t-1}^G + B_{t-3,t-1}^G + \ldots\right]$ accumulates by

$$B_t^G = N_t^G + (1-\mu) B_{t-1}^G, \quad N_t^G \equiv B_{t,t}^G. \tag{41}$$

In a SS, new debt replaces old debt at a rate μ, giving turnover $N^G = \mu B^G$.

The government raises tax revenue T_t and spends on productive services $P_t G_t$.[5] At the end of t, it issues new debt N_t^G at a price Q_t. The budget constraint restricts outflows (interest on old debt plus repayment) to inflows (new debt and *net* taxes, net of public spending) by

$$\begin{aligned} \bar{\imath} B_{t-1}^G + \mu B_{t-1}^G &= Q_t N_t^G + S_t^G, \quad S_t^G \equiv T_t - P_t G_t - E_t, \\ B_t^G &= (1+\bar{\imath}) B_{t-1}^G + (1-Q_t) N_t^G - S_t. \end{aligned} \tag{42}$$

Using (41) to replace $(1-\mu) B_{t-1}^G = B_t^G - N_t^G$ gives the dynamic budget constraint. If the required market return exceeds the coupon rate, long-term bonds sell at a discount relative to one period bonds ($Q_t < 1$). The lower revenue inflates end of period debt, if N_t^G is not adjusted. One period bonds ($\mu = 1$) issued at rate $\bar{\imath} = i_{t-1}^g$ give $Q_t = 1$. The budget reduces to the standard form $B_t^G = \left(1 + i_{t-1}^g\right) B_{t-1}^G - S_t$.

In contrast to the quantity (42), the market value of debt is forward looking. Multiply bond pricing (40) by B_t^G and get $(1+i_t^g) Q_t B_t^G = (\bar{\imath} + \mu) B_t^G + (1-\mu) B_t^G Q_{t+1}$. Substituting for $(1-\mu) B_t^G = B_{t+1}^G - N_{t+1}^G$ and noting the budget in (42) gives

$$(1+i_t^g) Q_t B_t^G = S_{t+1}^G + Q_{t+1} B_{t+1}^G. \tag{43}$$

Solving forward gives the intertemporal budget constraint which requires that the current value of debt is backed up by a present value of future surpluses.

[5] Producing public services uses labor and capital. Assuming the same factor intensity as in private production, we can model this as demand for domestic output only, without imports.

Avoiding unstable debt requires a consolidation policy. We assume that the government targets a long-run debt to GDP ratio $\bar{b}^g$ and aims to approach the long-run ratio with a desired adjustment speed, $Q_t B_t^G \to \bar{b}^g P_t Y_t$. We thus specify a policy rule for the 'structural' component $\tilde{S}_t^G$ of the primary surplus S_t^G which excludes any temporary surprise expenditures or revenues. Indeed, the Maastricht rules restrict the structural rather than the actual deficit, and also specify a long-run debt to GDP ratio $\bar{b}^g$. The parameter γ^g determines how fast debt is reduced (or increased) to reach the long-run target.

The consolidation policy thus specifies a structural surplus

$$\tilde{S}_t^G = (\bar{\imath} + \mu + (1-\mu) Q_t - \gamma^g Q_t) B_{t-1}^G - (1-\gamma^g) \bar{b}^g P_t Y_t, \quad 0 < \gamma^g < 1. \tag{44}$$

To motivate the consolidation rule, use $B_t^G = N_t^G + (1-\mu) B_{t-1}^G$ to replace new bond issues N_t^G in the primary surplus in (42), $Q_t B_t^G = (\bar{\imath} + \mu + (1-\mu) Q_t) B_{t-1}^G - S_t^G$. Absent fiscal shocks, structural and actual surpluses are identical, $S_t^G = \tilde{S}_t^G$. The end of period value of sovereign debt then exclusively depends on the target surplus $\tilde{S}_t^G$. Replacing $\tilde{S}_t^G$ by the policy rule results in $Q_t B_t^G = \gamma^g Q_t B_{t-1}^G + (1-\gamma^g) \bar{b}^g P_t Y_t$. When all variables become stationary, the stabilization rule also makes debt converge to $QB^G \to \bar{b}^g PY$.

The actual surplus deviates from the structural surplus when there are unexpected shocks. Spending policies and required tax revenues T_t are

$$\begin{aligned} P_t G_t &= \bar{g} \cdot P_t Y_t - \xi^g \cdot \tilde{S}_t^G + \varepsilon_t^G, \\ E_t &= \bar{e} \cdot w_t^L L_t - \xi^e \cdot \tilde{S}_t^G + \varepsilon_t^E, \\ T_t &= \bar{g} \cdot P_t Y_t + \bar{e} \cdot w_t^L L_t + (1 - \xi^g - \xi^e) \cdot \tilde{S}_t^G. \end{aligned} \tag{45}$$

Productive spending consists of a normal level $\bar{g} P_t Y_t$, reduced by spending cuts to finance a share ξ^g of the required primary surplus. Social spending reflects a normal replacement rate $\bar{e}$ of wage earnings. Spending cuts must contribute a share ξ^e to budget consolidation. Spending shocks ε_t^G and ε_t^E as well as unexpected subsidies to banks T_t^b are not immediately financed with taxes but raise next period's debt and are consolidated later on. In consequence, the required tax revenue T_t finances only the structural part of public spending, $\bar{g}P_tY_t + \bar{e}w_t^L L_t$, plus additional tax increases $(1-\xi^g-\xi^e)\,\tilde{S}_t^G$ needed to reduce public debt. The parameters ξ^e and ξ^g determine whether consolidation policy is tax or expenditure based. If ξ^e and ξ^g are low, most of budget consolidation is tax based while high values indicate budget consolidation with spending cuts.

To see how unconsolidated fiscal shocks affect debt dynamics, substitute tax and spending rules (45) into the actual primary surplus (42),

$$S_t^G = \tilde{S}_t^G - \varepsilon_t^G - \varepsilon_t^E. \tag{46}$$

In the absence of shocks, $S_t^G = \tilde{S}_t^G$, which leads to stable debt as in (42). Any unexpected spending first raises public debt before it gets consolidated in future periods.[6]

When raising ξ^g and ξ^e, the government shifts budget consolidation from tax increases to spending cuts. Our approach links to research by Alesina et al. (2015) on the effectiveness of tax versus spending based budget consolidation. In the spirit of Barro (1990), we assume that a higher stock of productive infrastructure K_t^G (broadly defined) boosts factor productivity by Z_t. The public capital stock accumulates by

$$K_t^G = G_t + (1-\delta^g)\,K_{t-1}^G, \quad Z_t = (1-\rho)\,\bar{Z}\left(1+\sigma^z\left(K_{t-1}^G-\bar{K}^G\right)/\bar{K}^G\right) + \rho Z_{t-1} + \varepsilon_t^Z. \tag{47}$$

Higher tax rates discourage labor supply and investment, thereby reducing growth. Public revenues stem from taxing wages, profits and consumption at rates τ_t, τ_t^k and τ_t^c, respectively. Given the profit tax base T_t^k, tax revenue is

$$\begin{aligned} T_t &= \tau_t \cdot w_t^L L_t + \tau_t^k \cdot T_t^k + \tau_t^c \cdot \bar{P}_t\bar{C}_t - T_t^l, \\ T_t^l &= \left(1-\rho^T\right)\bar{t}^l P_tY_t + \rho^T T_{t-1}^l + \varepsilon_t^T, \\ \tau_t &= t_t^s \cdot \bar{\tau}, \quad \tau_t^k = t_t^s \cdot \bar{\tau}^k, \quad \tau_t^c = t_t^s \cdot \bar{\tau}^c. \end{aligned} \tag{48}$$

Revenue shrinks by tax base erosion, leading to tax losses T_t^l. We assume that the tax yield is reduced by $\bar{t}^l$% of GDP in the long-run. More tax base erosion raises tax rates and

[6] Substitute (46) into (43), $Q_tB_t^G = \left(1+i_{t-1}^g\right)Q_{t-1}B_{t-1}^G - \tilde{S}_t^G + \varepsilon_t^G + \varepsilon_t^E + T_t^b$. The consolidation rule (44) thus results in actual debt dynamics of $Q_tB_t^G = \gamma^g Q_{t-1}B_{t-1}^G + (1-\gamma^g)\,\bar{b}^g P_tY_t + \varepsilon_t^G + \varepsilon_t^E + T_t^b$ which eventually converges with speed γ^g to the long-run target when the transitory parts vanish.

slows down growth. Finally, we introduce a scaler t_t^s to implement tax based consolidation. Initially, $t^s = 1$. When the government needs more revenue, it may scale up all tax rates by t_t^s. A change in the tax structure, e.g., a shift from income to consumption taxes, could also be implemented by first reducing the parameter $\bar{\tau}$ and then computing the budget balancing increase in t_t^s.

To sum up, consolidation policy determines the structural surpluses $\tilde{S}_t^G$ in (44) which results in an actual surplus S_t^G as in (46). To support the required tax revenue in (45) then requires to appropriately scale tax rates. The budget (42) finally determines new bond issues N_t^G, leading to end of period debt B_t^G by (41). Sovereign debt B_t^G is partly held by banks and partly traded on the capital market. Banks acquire a quantity N_t^g of newly issued bonds, giving total bond holdings of B_t^g. The rest is placed on the capital market. Private investors are willing to hold any residual quantity $B_t = B_t^G - B_t^g$, as long as bonds offer the required return i_t^g.

1.6 Banking Sector

Bond Trading: Banks are price takers on bond markets. The government issues a share $1 - \tilde{s}^b$ of debt on bond markets and sells the remainder to banks, giving bond holdings $B_t^g = \tilde{s}^b B_t^G$ of banks. Banks can trade bonds at a price Q_t independent of public debt management. The stock of bonds $B_t^g = B_{t,t}^g + B_{t-1,t}^g + B_{t-2,t}^g + \ldots$ consists of many vintages. At any date, the sovereign repays a fraction μ. Banks may also sell a common fraction δ, and are left with a share $1 - \mu - \delta$. Vintage s shrinks to $B_{s,t}^g = (1 - \mu - \delta)\, B_{s,t-1}^g$, so that $B_t^g = B_{t,t}^g + (1 - \mu - \delta)\left(B_{t-1,t-1}^g + B_{t-2,t-1}^g + \ldots\right)$. Using $N_t^g = B_{t,t}^g$,[7]

$$B_t^g = N_t^g + (1 - \mu - \delta)\, B_{t-1}^g, \quad V_t^g = Q_t N_t^g + (1 - \mu - \delta)\, V_{t-1}^g, \quad 0 \leq \delta \leq 1 - \mu. \tag{49}$$

Similarly, $V_t^g = Q_t B_{t,t}^g + Q_{t-1} B_{t-1,t}^g + Q_{t-2} B_{t-2,t}^g + \ldots$ is the book value of bonds reflecting acquisition costs. Using again $B_{s,t}^g = (1 - \mu - \delta)\, B_{s,t-1}^g$ gives the second equation. Two

[7] New bond purchases are residually determined by $N_t^g = B_t^g - (1 - \mu - \delta)\, B_{t-1}^g$ with B_t^g and B_{t-1}^g being a fixed share $\tilde{s}^b$ of total stocks B_t^G.

limiting cases are informative. If $\delta = 0$, there would be no trading on the capital market. Banks would simply hold on to bonds until maturity and receive repayment at face value. Turnover of bond holdings and, in turn, convergence of market and book values would be very slow. If $\delta = 1-\mu$, banks would replace in each period the entire stock of outstanding bonds by new vintages. In this case, $B_t^g = N_t^g$ and $V_t^g = Q_t B_t^g$, so that market and book values are brought in line instantaneously. The bank would realize large losses when selling of bonds with high book value at a low price.

Balance Sheet: Assets and liabilities are credit B_t^l, bond holdings V_t^g, equity E_t^b and deposits D_t. The balances sheet is $B_t^l + V_t^g = E_t^b + D_t$. Stocks accumulate by

$$\begin{aligned} &B_t^l = \left(1-\delta^k - s_t\right) B_{t-1}^l + N_t^l, \quad D_t = D_{t-1} + N_t^d, \\ &V_t^g = Q_t N_t^g + (1-\mu-\delta) V_{t-1}^g, \quad E_t^b = E_{t-1}^b + N_t^b - \left(\delta^k + s_t\right) B_{t-1}^l - (\mu+\delta) V_{t-1}^g. \end{aligned} \tag{50}$$

While a share δ^k of loans gets regularly repaid, a share s_t is bad. Banks must thus write off part of the loans. In total, the loan volume and bank equity shrink by $\left(\delta^k + s_t\right) B_{t-1}^l$. The balance sheet thus gives a flow constraint $\nabla B_t^l + \nabla V_t^g = \nabla E_t^b + \nabla D_t$, or

$$B_t^l + V_t^g = E_t^b + D_t \quad \Rightarrow \quad N_t^l + Q_t N_t^g = N_t^b + N_t^d. \tag{51}$$

In bankruptcy, a bank seizes all assets and can recover a share $1-\ell_t$ of the non-performing loan.[8] In period t, net losses on bad loans are $\ell_t s_t B_{t-1}^l$. Liquidation revenues show up in the bank's profit statement below. New loans N_t^l only go to viable projects, thereby making loan accumulation of banks and (surviving) businesses identical.

The banking sector budget records *inflows* consisting of loan interest earnings $i_t^l B_{t-1}^l$, repayment $\delta^k B_{t-1}^l$, liquidation revenues $(1-\ell_t) s_t B_{t-1}^l$ in t on bad loans, coupon interest plus repayment at face value $(\bar{\imath}+\mu) B_{t-1}^g$ of bond holdings, revenues from selling off an additional δB_{t-1}^g at a price Q_t, and new deposits N_t^d. *Outflows* are interest $i_{t-1}^d D_{t-1}$ on deposits, new lending N_t^l, bond purchases $Q_t N_t^g$ and dividends χ_t,

$$\begin{aligned} &: \quad i_t^l B_{t-1}^l + \delta^k B_{t-1}^l + (1-\ell_t) s_t B_{t-1}^l + (\bar{\imath}+\mu) B_{t-1}^g + \delta Q_t B_{t-1}^g + N_t^d \\ &: \quad\quad = i_{t-1}^d D_{t-1} + N_t^l + Q_t N_t^g + \chi_t^b. \end{aligned} \tag{52}$$

[8] The dating is discussed in detail in equation (69) below. Bank losses depend on firm leverage in $t-1$.

Use (51) to replace D_{t-1} and N_t^l. Define $\pi_t^g \equiv (\bar{\imath} + \mu + \delta Q_t) B_{t-1}^g - i_{t-1}^d V_{t-1}^g$ and get

$$N_t^b + \chi_t^b = \pi_t^b \equiv \left(i_t^l + (1 - \ell_t) s_t + \delta^k - i_{t-1}^d\right) B_{t-1}^l + i_{t-1}^d E_{t-1}^b + \pi_t^g. \tag{53}$$

The interest margin on bonds is captured by π_t^g. Bond earnings $(\bar{\imath} + \mu + \delta Q_t) B_{t-1}^g$ reflect coupon interest, repayment at maturity and revenues from selling off prior to maturity. The book value V_{t-1}^g must be backed up by deposits and creates interest cost $i_{t-1}^d V_{t-1}^g$.

Capital Constraint: Banks must satisfy minimum capital standards. Including buffers, equity must be at least κ^B of loans and κ^G of sovereign bonds,

$$E_t^b \geq \kappa^B B_t^l + \kappa^G V_t^g. \tag{54}$$

Once equity is fixed, deposits follow residually from the balance sheet identity. Anticipating a binding constraint, deposits are

$$D_t = \left(1 - \kappa^B\right) B_t^l + \left(1 - \kappa^G\right) V_t^g. \tag{55}$$

Optimization: Owners value steady dividends close to a benchmark $\bar{\chi}^b$. Deviations create convex increasing adjustment costs $z\left(\chi_t^b\right)$ in terms of output. When cutting dividends below $\bar{\chi}^b$, banks meet progressive resistance. Owners are also concerned about an erosion of equity capital when a bank pays out too much:

$$z\left(\chi_t^b\right) = \frac{1}{2}\psi^b \left(\chi_t^b - \bar{\chi}^b\right)^2. \tag{56}$$

When banks pay a *gross* dividend χ_t^b, investors receive a *net* dividend $\chi_t - z_t \bar{P}_t$, net of agency costs measured in terms of output. Bank valuation is

$$V_t^b = \chi_t^b - z_t \bar{P}_t + V_{t+1}^b / \left(1 + i_t^b\right). \tag{57}$$

Banks invest in loans and bonds and choose retained earnings subject to the financing constraint (54). Given the value function $V_t^b = V\left(B_{t-1}^l, E_{t-1}^b\right)$, they must solve[9]

$$\begin{aligned}
V_t^b &= \max_{N_t^l, N_t^b} \chi_t^b - z\left(\chi_t^b\right)\bar{P}_t + \frac{V_{t+1}^b + \mu_t^e\left[E_t^b - \kappa^B B_t^l - \kappa^G V_t^g\right]}{1+i_t^b} \\
s.t. &: \chi_t^b = \left(i_t^l + (1-\ell_t)s_t + \delta^k - i_{t-1}^d\right)B_{t-1}^l + i_{t-1}^d E_{t-1}^b - N_t^b + \pi_t^g, \\
&: B_t^l = N_t^l + \left(1 - \delta^k - s_t\right)B_{t-1}^l, \\
&: E_t^b = E_{t-1}^b + N_t^b - \left(\delta^k + s_t\right)B_{t-1}^l - (\mu+\delta)V_{t-1}^g.
\end{aligned} \tag{58}$$

Use $dV_t^b/B_{t-1}^l \equiv \lambda_t^l$, and $dV_t^b/E_{t-1}^b \equiv \lambda_t^e$. Get optimality and envelope conditions

$$\begin{aligned}
N_t^l &: 0 = \frac{\lambda_{t+1}^l - \kappa^B\mu_t^e}{1+i_t^b}, \\
N_t^b &: 0 = -\left(1 - z_t'\bar{P}_t\right) + \frac{\lambda_{t+1}^e + \mu_t^e}{1+i_t^b}, \\
B_{t-1}^l &: \lambda_t^l = \left(1 - z_t'\bar{P}_t\right)\left(i_t^l + (1-\ell_t)s_t + \delta^k - i_{t-1}^d\right) \\
&: +\left(1 - \delta^k - s_t\right)\frac{\lambda_{t+1}^l - \kappa^B\mu_t^e}{1+i_t^b} - \left(\delta^k + s_t\right)\frac{\lambda_{t+1}^e + \mu_t^e}{1+i_t^b}, \\
E_{t-1}^b &: \lambda_t^e = \left(1 - z_t'\bar{P}_t\right)i_{t-1}^d + \frac{\lambda_{t+1}^e + \mu_t^e}{1+i_t^b}.
\end{aligned} \tag{59}$$

Lending: Combine with optimality conditions to rewrite (iii) and (iv),

$$\lambda_t^l = \left(1 - z_t'\bar{P}_t\right)\left(i_t^l - \ell_t s_t - i_{t-1}^d\right), \quad \lambda_t^e = \left(1 - z_t'\bar{P}_t\right)\left(1 + i_{t-1}^d\right). \tag{60}$$

Use λ_{t+1}^e in (ii) to get $\mu_t^e = \left(1 - z_t'\bar{P}_t\right)\left(1 + i_t^b\right) - \left(1 - z_{t+1}'\bar{P}_{t+1}\right)\left(1 + i_t^d\right)$. The shadow price is positive in a neighborhood of a SS, $\mu^e = \left(i^b - i^d\right)\left(1 - z'\bar{P}\right) > 0$, since $i^b > i^d$ by the equity premium and z' roughly zero. The financing constraint is binding. To relate the shadow price to the banks' earnings potential, we substitute λ_{t+1}^l into (i),

$$\mu_t^e = \lambda_{t+1}^l/\kappa^B = \left(1 - z_{t+1}'\bar{P}_{t+1}\right)\left(i_{t+1}^l - \ell_{t+1}s_{t+1} - i_t^d\right)/\kappa^B. \tag{61}$$

With a binding finance constraint, lending is a fixed leverage of free equity,

$$B_t^l = \left(E_t^b - \kappa^G V_t^g\right)/\kappa^B. \tag{62}$$

[9] Bond holdings are not optimized. All terms relating to public debt are thus exogenous to the problem.

Dividends: Divide (59.i) by κ^B and add to (ii). Define a combined shadow price, $\lambda_{t+1}^b \equiv \lambda_{t+1}^e + \lambda_{t+1}^l/\kappa^B$, and Tobin's Q on retained earnings, $Q_t^b \equiv \lambda_{t+1}^b / \left(1 + i_t^b\right)$. Dividend policy follows from $1 - z_t' \bar{P}_t = Q_t^b$,

$$\begin{aligned} \chi_t^b &= \bar{\chi}^b + \left(1 - Q_t^b\right) / \left(\psi^b \bar{P}_t\right), \quad Q_t^b \equiv \lambda_{t+1}^b / \left(1 + i_t^b\right), \quad \lambda_{t+1}^b \equiv \lambda_{t+1}^e + \lambda_{t+1}^l / \kappa^B, \\ \lambda_t^b &= \left[1 + i_{t-1}^d + \left(i_t^l - \ell_t s_t - i_{t-1}^d\right) / \kappa^B\right] \lambda_{t+1}^b / \left(1 + i_t^b\right). \end{aligned} \tag{63}$$

To obtain the second equation, divide (59.iii) by κ^B, add to (iv), note $\mu_t^e = \lambda_{t+1}^l / \kappa^B$ by (61) and apply the definition $\lambda_t^b \equiv \lambda_t^e + \lambda_t^l / \kappa^B$. Finally using $1 - z_t' \bar{P}_t = \lambda_{t+1}^b / \left(1 + i_t^b\right)$ and rearranging gives the result.

Using $\chi_t^b + N_t^b = \pi_t^b$, retained earnings are $N_t^b = \pi_t^b - \bar{\chi}^b + \left(Q_t^b - 1\right) / \left(\psi^b P_t\right)$. If Tobin's $Q_t^b > 1$, equity is more valuable than other investments. Retained earnings increase.

Loan Pricing: By (63.ii), $\tilde{i}_t^b \equiv \left(1 + i_t^b\right) \lambda_t^b / \lambda_{t+1}^b - 1 = i_{t-1}^d + \left(i_t^l - \ell_t s_t - i_{t-1}^d\right) / \kappa^B$, where $\tilde{i}_t^b$ is an effective cost of equity. We obtain

$$i_t^l = \kappa^B \cdot \tilde{i}_t^b + \left(1 - \kappa^B\right) \cdot i_{t-1}^d + \ell_t s_t, \quad \tilde{i}_t^b \equiv \left(1 + i_t^b\right) \lambda_t^b / \lambda_{t+1}^b - 1. \tag{64}$$

The loan rate must match funding costs plus $\ell_t s_t$ to cover marginal credit losses. In a SS, $\lambda_t^b = \lambda_{t+1}^b$ implies $\tilde{i}^b = i^b$, giving

$$i^l = \kappa^B \cdot i^b + \left(1 - \kappa^B\right) \cdot i^d + \ell s, \quad \tilde{i}^b = i^b. \tag{65}$$

Shocks: The share of bad loans follows an autoregressive process

$$s_t = (1 - \rho^s) \left(\bar{s} + \sigma^s \cdot \left(\bar{Y} - Y_t\right) / \bar{Y}\right) + \rho^s s_{t-1} + \varepsilon_t^s. \tag{66}$$

The semi-elasticity σ^s determines the response to an output gap, $ds = -\sigma^s \cdot dY_t / \bar{Y}$. When the economy returns to the ISS, $Y_t = \bar{Y}$ and $\varepsilon_t^s = 0$, giving $s = \bar{s}$.

1.7 General Equilibrium

Monetary Policy: We analyze fluctuations around a steady state with constant money supply and zero inflation. We specify a policy rule as in Ascari and Ropele (2013) and

Sargent and Surico (2011),[10]

$$M_t^s = (1-\rho^m)\,\phi^m \bar{Y}_t \cdot \frac{\left(\bar{Y}_t/Y_t\right)^{\psi^{yi}}}{(1+\pi_t)^{\psi^\pi}} + \rho^m M_{t-1}^s + \varepsilon_t^m, \quad T_t^M = M_t^s - M_{t-1}^s. \tag{67}$$

Money supply consists of a trend and a cyclical component. Depending on parameters ψ^{yi} and ψ^π, monetary policy aims to dampen fluctuations around potential (trend) output $\bar{Y}_t$. If current output is below potential output, $Y_t < \bar{Y}_t$, money supply is increased by a factor $\left(\bar{Y}_t/Y_t\right)^{\psi^{yi}} > 1$, while the oppposite happens in a boom. Similarly, if actual inflation exceeds the trend rate ($\pi_t > 0$), money supply is scaled down by $1/\left(1+\pi_t\right)^{\psi^\pi} < 0$. The opposite happens in a recession where $\pi_t < 0$.

Potential output may change, reflecting the impact of structural change or fiscal policy. Money supply should change by the trend component $\phi^m \bar{Y}_t$ when potential and actual output are identical. Money supply thus accommodates any change in potential output along an equilibrium growth path $Y_t = \bar{Y}_t$. To allow for a change in trend growth (in levels), we model potential output as

$$\bar{Y}_t = (1-\rho^y)\,\bar{Y}_\infty + \rho^y \bar{Y}_{t-1} \quad \Rightarrow \quad \bar{Y}_t = (\rho^y)^t\,\bar{Y}_0 + (1-\rho^y)^t\,\bar{Y}_\infty. \tag{68}$$

In a FSS after structural change, actual and potential output are $\bar{Y}_\infty = Y_{ss} = Y$. The transition path is independent of any temporary output fluctuation. The transition path of potential output follows $\bar{Y}_t = \bar{Y}_\infty + (\rho^y)^t\left(\bar{Y}_0 - \bar{Y}_\infty\right)$. With $\rho^y < 1$, it starts from an initial value $\bar{Y}_0$ and converges to $\bar{Y}_\infty$ for $t \to \infty$. The half-life of adjustment is $t_{0.5} = \log(0.5)/\log(\rho^y)$. A half-life of 6 years or 24 quarters gives a root $\rho^y = \exp\left(\log(0.5)/t_{0.5}\right) \approx 0.9715$. In the absence of structural change, $\bar{Y}_0 = \bar{Y}_\infty = \bar{Y}_t$, so that potential output is completely flat.

Aggregate Income: Nominal GDP is $P_t Y_t$. Income losses due to liquidation are reflected in investment demand. Liquidation creates a market for used capital goods.

[10] With common monetary policy, output and inflation sensitivities are uniform, ψ^π and $\psi^{yi} = \psi^y$. With monetary autonomy, we allow Italy to choose a more aggressive policy with the output sensitivity being a factor $\psi^i > 1$ larger then in the rest of the Eurozone, $\psi^{yi} = \psi^i \psi^y$.

Supply $\left(1-\ell^k\right) s_t K_{t-1}$ depends on the liquidation rate. Demand for the composite good absorbs newly produced and used equipment. In addition, bank owners incur dividend adjustment costs z_t in terms of forgone consumption, $\bar{I}_t + J_t + z_t = \bar{I}_t^D + \left(1-\ell^k\right) s_t K_{t-1}$. Net demand $\bar{I}_t^D$ consists of domestic goods and imports (as in 4-5). The investment budget for new capital goods is $\bar{P}_t \bar{I}_t^D = P_t I_t + P_t^{ie} I_t^{ie} + P_t^{io} I_t^{io}$. We record

$$\bar{I}_t^D = \bar{I}_t + J_t + z_t - \left(1-\ell^k\right) s_t K_{t-1}, \quad \ell_t = 1 - \bar{P}_t \left(1-\ell^k\right) K_{t-1}/B_{t-1}^l. \tag{69}$$

The loss rates ℓ_t and ℓ^k on bad loans and on assets are linked. When liquidating $s_t K_{t-1}$ capital goods, banks can sell them at a discount on the market for used equipment and earn $\bar{P}_t \left(1-\ell^k\right) s_t K_{t-1}$. Liquidation revenues define the loan recovery rate $1-\ell_t$ by the budget $\left(1-\ell_t\right) s_t B_{t-1}^l = \bar{P}_t \left(1-\ell^k\right) s_t K_{t-1}$. Using $B_{t-1}^l = b^k \bar{P}_{t-1} K_{t-1}$, the liquidation budget is equivalent to $1-\ell_t = \frac{1-\ell^k}{b^k} \frac{\bar{P}_t}{\bar{P}_{t-1}}$. The loss and recovery rates reflect an equilibrium relationship and are taken as given by banks. Since ℓ^k is fixed, a higher firm leverage b^k reduces the recovery rate today. Quite intuitively, loan recovery improves when used equipment is sold at higher prices $\bar{P}_t$.

Apart from wages and interest on internationally traded bonds, households collect capital income χ_t^A including dividends from ownership of firms and banks (net of dividend adjustment costs) of χ_t^k and $\chi_t^b - z_t \bar{P}_t$; interest on deposits net of new deposit savings $i_{t-1}^d D_{t-1} - N_t^d$; and net earnings on sovereign bond holdings. Specifically, households absorb a quantity $B_t^h = B_t^G - B_t^g$ of bonds. Given a repayment rate μ, bond holdings change by $B_t^h = N_t^h + (1-\mu) B_{t-1}^h$. Investors receive income $(\bar{\imath} + \mu) B_{t-1}^h$ including coupon interest and repayment at rates $\bar{\imath}$ and μ, and spend $Q_t N_t^h$ to acquire new bonds where $N_t^h = N_t^G + \delta B_{t-1}^g - N_t^g$. In total,

$$\chi_t^A = \chi_t^k + \chi_t^b - z_t \bar{P}_t + \left(i_{t-1}^d D_{t-1} - N_t^d\right) + (\bar{\imath} + \mu) B_{t-1}^h - Q_t N_t^h. \tag{70}$$

Current Account: B_t^f is NFA ($B_t^f < 0$ is foreign debt) in domestic currency,

$$B_t^f = (1 + i_{t-1}) B_{t-1}^f + TB_t, \quad TB_t = P_t E_t^x - P_t^{ie}\left(C_t^{ie} + I_t^{ie}\right) - P_t^{io}\left(C_t^{io} + I_t^{io}\right). \tag{71}$$

The trade surplus TB_t, in domestic currency, is equal to the value of exports minus imports. Exports reflect import demand of other regions. If Italy accumulates net debt $-B^f$, it must run a trade surplus to pay for interest, $TB = -iB^f$ in a SS.

In (71), net debt is denominated in domestic currency (Lira), and is exclusively held by Eurozone investors.[11] There is exchange rate risk. If an EZ saver invests 1 Euro at home, she earns interest i_t^e which accrues next period, giving $1 + i_t^e$. If she invests 1 Euro in the Italian bond at the end of t, she gets e_t^{ie} Lire which yield $(1 + i_t)\, e_t^{ie}$ next period and are converted back at a rate $1/e_{t+1}^{ie}$. Standard interest rate parity prevents arbitrage, $1 + i_t^e = (1 + i_t)\, e_t^{ie}/e_{t+1}^{ie}$. However, when countries are subject to default risk, investors request a premium $\theta_t^f \geq 1$. It is assumed to rise in line with debt to GDP ratios beyond some benchmark, $b_t^f \geq \bar{b}^f$. Modified interest parity requires

$$(1 + i_t)\, e_t^{ie}/e_{t+1}^{ie} = (1 + i_t^e)\, \theta_t^f. \tag{72}$$

The Italian bond yield in Euros must exceed the domestic return $1 + i_t^e$ by a factor θ_t^f. Assets are perfect substitutes up to the premium θ_t^f. When the country's debt ratio rises (or NFA fall below the benchmark $\bar{b}^f$), investors start to worry about solvency and ask for a higher premium. Conversely, the premium falls when a country accumulates NFA above the benchmark level, making it a safe haven. In addition, we introduce autoregressive shocks to capture sudden stop phenomena in foreign funding,

$$\theta_t^f = \left(1 - \rho^f\right) \left[1 + \gamma \left(e^{\bar{b}^f - b_t^f} - 1\right)\right] + \rho^f \theta_{t-1}^f + \varepsilon_t^f, \quad b_t^f \equiv B_t^f / \left(P_t Y_t\right). \tag{73}$$

In a SS, exchange rates are constant and $i = i^e = 1/\beta$ to keep consumption flat. The country premium must disappear, $\theta^f \to 1$, so that $b_t^f \to \bar{b}^f$. The model thus explains fluctuations around a stationary foreign debt to GDP ratio. The sensitivity of the country premium to the NFA position assures stability of savings in an open economy.

Market Clearing: Equilibrium requires market clearing

$$Y_t = C_t + G_t + I_t + E_t^x, \quad A_t = B_t^f, \quad \bar{M}_t = M_t^s/\bar{P}_t. \tag{74}$$

[11] Our focus is on Italy and the Eurozone. We thus do not explain capital flows with RoW.

Demand for home goods stems from consumption, investment and exports. Households absorb bank deposits and government bonds and hold shares in firms and banks and receive net capital income χ_t^A from these sources, as in (70). Any residual savings A_t are invested in net foreign assets (negative foreign debt). The last condition relates demand for real money balances to money supply, $\bar{M}_t = M_t^s/\bar{P}_t$. Market clearing fixes the price level $\bar{P}_t = M_t^s/\bar{M}_t$. There is no separate condition for labor market clearing since each household type j is a 'local' monopolist and serves the entire market, $N_{j,t}\bar{N} = L_{j,t}$.

Lemma 1 ***Walras' Law:*** *Excess demands are related by*

$$(A_t - B_t^f) - (1 + i_{t-1})(A_{t-1} - B_{t-1}^f) + (M_t - M_t^s) + P_t(C_t + G_t + I_t + E_t^x - Y_t) = 0. \quad (75)$$

Proof. To compute χ_t^A in (70), substitute dividends χ_t^k and χ_t^b from (27,52). Use $B_{t-1}^G = B_{t-1}^h + B_{t-1}^g$ and $N_t^h = N_t^G + \delta B_{t-1}^g - N_t^g$ and consolidate terms. Next, use the fiscal budget (42) to replace new bond issues $Q_t N_t^G$ and substitute for tax revenue T_t as in (48). Finally, we use $(1-\ell_t)\, s_t B_{t-1}^l = \bar{P}_t\left(1-\ell^k\right) s_t K_{t-1} = \bar{P}_t\left(\bar{I}_t + J_t + z_t - \bar{I}_t^D\right)$ which holds by the realized liquidation budget in (69),

$$\chi_t^A = P_t(Y_t - G_t) - \bar{P}_t\bar{I}_t^D - E_t - (1-\tau_t)\, w_t^L L_t - T_t^l + \tau_t^c \bar{P}_t \bar{C}_t. \quad \text{(i)}$$

Use $\int_0^1 w_{j,t} N_{j,t}\bar{N} dj = w_t^L L_t$ and substitute χ_t^A and $T_t^M = M_t^s - M_{t-1}^s$ into the household budget (13). Market clearing $M_{t-1} = M_{t-1}^s$ holds identically in $t-1$, so that

$$A_t + (M_t - M_t^s) = (1 + i_{t-1})\, A_{t-1} + \left[P_t(Y_t - G_t) - \bar{P}_t\left(\bar{C}_t + \bar{I}_t^D\right)\right]. \quad \text{(ii)}$$

Rearrange (ii), $P_t Y_t + i_{t-1}A_{t-1} = \left[P_t G_t + \bar{P}_t\left(\bar{C}_t + \bar{I}_t\right)\right] + [(A_t - A_{t-1}) + (M_t - M_t^s)]$, to find that GNP is domestic absorption plus savings (income expenditure identity). Expand square bracket in (ii) by exports $P_t E_t^x$, substitute budgets $\bar{P}_t\bar{I}_t^D = P_t I_t + P_t^{ie} I_t^{ie} + P_t^{io} I_t^{io}$ and $\bar{P}_t\bar{C}_t$, and note the definition of the trade balance in (71) to obtain

$$P_t(Y_t - G_t) - \bar{P}_t\left(\bar{C}_t + \bar{I}_t^D\right) = P_t(Y_t - C_t - G_t - I_t - E_t^x) + TB_t. \quad (76)$$

With output market clearing, the trade surplus is GDP minus domestic absorption. Combine (ii) and (76) to eliminate absorption and use the current account (71) to replace TB_t.

This proves the result in (75). By Walras' Law, one of the market clearing conditions is redundant. Note that $A_{t-1} - B^f_{t-1} = 0$ by previous period equilibrium. ■

2 The World Economy

2.1 Rest of Eurozone

Households: Eurozone modeling is extremely simple. We postulate an autoregressive income process for Eurozone GDP,

$$Y^e_t = (1-\rho)\,\bar{Y}^e + \rho Y^e_{t-1} + \varepsilon^{Y,e}_t. \tag{77}$$

Households collect income, consume and save. Savings in domestic assets yield a return i^e_{t-1}. Italian bonds yield a country premium. To the extent that households are invested in Italian bonds, they collect differential interest earnings $\chi^{f,e}_t$ (defined below with the current account). Wealth accumulation in Euros thus follows

$$A^e_t = \left(1 + i^e_{t-1}\right) A^e_{t-1} + \chi^{f,e}_t + P^e_t Y^e_t + T^{M,e}_t - \left(M^e_t - M^e_{t-1}\right) - \bar{P}^e_t \bar{C}^e_t. \tag{78}$$

Agents spend $\bar{P}^e_t \bar{C}^e_t = P^e_t C^e_t + P^{ei}_t C^{ei}_t + P^{eo}_t C^{eo}_t$ where $P^{ei}_t = P^i_t / e^{ie}_t$ and $P^{eo}_t = P^o e^{eo}_t$ are local demand prices in Euros. One Dollar is worth e^{eo}_t Euros. If a RoW good costs P^o Dollars, the price in Euros is $P^{eo}_t = P^o e^{eo}_t$. EZ consumers demand three goods, subject to $\bar{C}^e_t = \left[\sum_r \left(s^{er}_r\right)^{1/\sigma^r} \left(C^{er}_t\right)^{(\sigma^r-1)/\sigma^r}\right]^{\sigma^r/(\sigma^r-1)}$. Minimizing expenditure yields a price index $\bar{P}^e_t$, demand C^{er}_t for goods of region r, and total spending $\bar{P}^e_t \bar{C}^e_t$,

$$C^{er}_t = s^{er}_r \left(\bar{P}^e_t / P^{er}_t\right)^{\sigma^r} \bar{C}^e_t, \quad \bar{P}^e_t = \left[\sum_r s^{er}_r \left(P^{er}_t\right)^{1-\sigma^r}\right]^{1/(1-\sigma^r)}. \tag{79}$$

Preferences for consumption $\bar{C}^e_t$ and real money balances $\bar{M}^e_t$ are

$$V^e_t = u^e\left(\bar{C}^e_{t+s}, \bar{M}^e_{t+s}\right) + \bar{\beta} E_t V^e_{t+1}, \quad u^e_t = \frac{\left(\bar{C}^e_t\right)^{1-1/\sigma^c}}{1-1/\sigma^c} + \left(m^e_t\right)^{1/\sigma^m} \frac{\left(\bar{M}^e_t\right)^{1-1/\sigma^m}}{1-1/\sigma^m}. \tag{80}$$

Real money demand is s.t. shocks $m^e_t = (1-\rho^m)\,\bar{m}^e + \rho^m m^e_{t-1} + \varepsilon^{m,e}_t$. We record

$$u^e_{\bar{C},t} = 1/\left(\bar{C}^e_t\right)^{1/\sigma^c}, \quad u^e_{\bar{M},t} = \left(m^e_t / \bar{M}^e_t\right)^{1/\sigma^m}, \quad u^e_{\bar{M},t}/u^e_{\bar{C},t} = \left(m^e_t/\bar{M}^e_t\right)^{1/\sigma^m} \left(\bar{C}^e_t\right)^{1/\sigma^c}.$$

Solution: The Bellman problem to be solved is

$$\begin{aligned} V^e\left(A_{t-1}^e, M_{t-1}^e\right) &= \max_{\bar{C}_t^e, M_t^e} \; u\left(\bar{C}_t^e, \bar{M}_t^e\right) + \bar{\beta} E_t V^e\left(A_t^e, M_t^e\right) \quad s.t. \\ A_t^e &= \left(1+i_{t-1}^e\right) A_{t-1}^e + \chi_t^{f,e} + P_t^e Y_t^e + T_t^{M,e} - \left(M_t^e - M_{t-1}^e\right) - \bar{P}_t^e \bar{C}_t^e. \end{aligned}$$

Define $\lambda_{t+1}^e \equiv dV_{t+1}^e / dA_t^e$ and $\lambda_t^{M,e} \equiv dV_t^e / dM_{t-1}^e$. Using $M_t^e = \bar{M}_t^e \bar{P}_t^e$, optimality conditions for $\bar{C}_t^e, \bar{M}_t^e$ and envelope conditions for A_{t-1}^e, M_{t-1}^e are

$$\begin{aligned} \bar{C}_t^e &: u_{\bar{C},t}^e = \bar{\beta} E_t \lambda_{t+1}^e \bar{P}_t^e, \\ M_t^e &: u_{\bar{M},t}^e = \bar{\beta} E_t \left(\lambda_{t+1}^e - \lambda_{t+1}^{M,e}\right) \bar{P}_t^e, \\ A_{t-1}^e &: \lambda_t^e = \bar{\beta} E_t \lambda_{t+1}^e \left(1+i_{t-1}^e\right), \\ M_{t-1}^e &: \lambda_t^{M,e} = \bar{\beta} E_t \lambda_{t+1}^e. \end{aligned} \tag{81}$$

Shift forward (81.iii) by one period, multiply by $\bar{\beta} E_t \bar{P}_t^e$, rearrange terms to use (i) on both sides and get the Euler condition,

$$u_{\bar{C},t}^e = E_t \bar{\beta}\left(1+r_t^e\right) \cdot u_{\bar{C},t+1}^e \quad \Rightarrow \quad 1 = \left(E_t \bar{\beta}\left(1+r_t^e\right)\right)^{\sigma^c} \left(\bar{C}_t^e / \bar{C}_{t+1}^e\right). \tag{82}$$

The real interest rate (depending on *expected* inflation) is

$$1 + r_t^e = \left(1+i_t^e\right) / \left(1+\pi_t^e\right), \quad 1+\pi_t^e = \bar{P}_{t+1}^e / \bar{P}_t^e. \tag{83}$$

By (iii-iv), $\lambda_t^{M,e} = \lambda_t^e / \left(1+i_{t-1}^e\right)$. Using this in (ii), combining with (i) gives the tangency condition for money demand,

$$\frac{u_{\bar{M},t}^e}{u_{\bar{C},t}^e} = \frac{i_t^e}{1+i_t^e} \quad \Rightarrow \quad \bar{M}_t^e = m_t^e \cdot \left[\frac{1+i_t^e}{i_t^e}\left(\bar{C}_t^e\right)^{1/\sigma^c}\right]^{\sigma^m}. \tag{84}$$

Equilibrium: The policy rule for money supply is

$$\begin{aligned} M_t^{s,e} &= \left(1-\rho^m\right) \phi^{m,e} \bar{Y}_t^e \cdot \frac{\left(\bar{Y}_t^e / Y_t^e\right)^{\psi^y}}{\left(1+\pi_t^e\right)^{\psi^\pi}} + \rho^m M_{t-1}^{s,e} + \varepsilon_t^{m,e}, \\ \bar{Y}_t^e &= \left(1-\rho^y\right) \bar{Y}_\infty^e + \rho^y \bar{Y}_{t-1}^e, \quad T_t^{M,e} = M_t^{s,e} - M_{t-1}^{s,e}. \end{aligned} \tag{85}$$

Potential output is parallel to (67), where $\bar{Y}_\infty^e = \bar{Y}^e$ is the long-run value in (77).

The rest of the EZ is a creditor country and issues net foreign assets $B_t^{f,e} > 0$. In buying Italian external debt, see (71), it earns a premium θ_t^f on cross-country investment, as discussed in (73). The NFA position is denominated in Euros and grows by

$$\begin{aligned} B_t^{f,e} &= \left(1+i_{t-1}^e\right) B_{t-1}^{f,e} + \chi_t^{f,e} + TB_t^e, \quad TB_t^e = P_t^e E_t^{x,e} - P_t^{ei} C_t^{ei} - P_t^{eo} C_t^{eo}, \\ &: \quad \chi_t^{f,e} = \left[(1+i_{t-1})\, e_{t-1}^{ie}/e_t^{ie} - \left(1+i_{t-1}^e\right)\right] B_{t-1}^{f,e}. \end{aligned} \tag{86}$$

Actual interest on holding Italian bonds is $(1+i_{t-1})\, e_{t-1}^{ie}/e_t^{ie}$. Foreign assets thus earn national interest i_{t-1}^e plus a differential reflecting cross-border investment risk. End of period wealth adds up to $\left(1+i_{t-1}^e\right) B_{t-1}^{f,e} + \chi_t^{f,e} = \left[(1+i_{t-1})\, e_{t-1}^{ie}/e_t^{ie}\right] B_{t-1}^{f,e}$.

With residual savings invested in Italian bonds $B_t^{f,e}$, market clearing conditions are

$$Y_t^e = C_t^e + E_t^{x,e}, \quad A_t^e = B_t^{f,e}, \quad \bar{M}_t^e = M_t^{s,e}/\bar{P}_t^e, \quad T_t^{M,e} = M_t^{s,e} - M_{t-1}^{s,e}. \tag{87}$$

Lemma 2 ***(Walras' Law)*** *Excess demands are related by*

$$\left(A_t^e - B_t^{f,e}\right) + \left(M_t^e - M_t^{s,e}\right) + P_t^e\left(C_t^e + E_t^{x,e} - Y_t^e\right) = 0. \tag{88}$$

Proof. Use $T_t^{M,e} = M_t^{s,e} - M_{t-1}^{s,e}$ in the household budget (78),

$$A_t^e + \left(M_t^e - M_t^{s,e}\right) - \left(M_{t-1}^e - M_{t-1}^{s,e}\right) = \left(1+i_{t-1}^e\right) A_{t-1}^e + \chi_t^{f,e} + P_t^e Y_t^e - \bar{P}_t^e \bar{C}_t^e. \tag{i}$$

Expand the r.h.s. by exports $P_t^e E_t^{x,e}$, substitute the budget $\bar{P}_t^e \bar{C}_t^e$ and the trade balance,

$$P_t^e Y_t^e - \bar{P}_t^e \bar{C}_t^e = P_t^e\left(Y_t^e - C_t^e - E_t^{x,e}\right) + TB_t^e. \tag{89}$$

Given output market clearing, the trade balance is GDP minus absorption. Substituting into (i) and using the current account (86) together with $\chi_t^{f,e}$ gives

$$\begin{aligned} &: \quad \left(A_t^e - B_t^{f,e}\right) + \left(M_t^e - M_t^{s,e}\right) + P_t^e\left(C_t^e + E_t^{x,e} - Y_t^e\right) \\ &= \quad \left(M_{t-1}^e - M_{t-1}^{s,e}\right) + \left(1+i_{t-1}^e\right)\left(A_{t-1}^e - B_{t-1}^{f,e}\right) = 0. \end{aligned} \tag{ii}$$

Since previous period market clearing holds by definition, this yields the result in (88). By Walras' Law, one of the market clearing conditions is redundant. ■

2.2 Rest of the World

RoW consists of other countries (indexed by o). Since our focus is on Italy and the Eurozone, we allow only for internal capital flows. Hence, net foreign assets of RoW are zero and trade is balanced. The foreign final good serves as *numeraire*, the local price is $P^o = 1$. We postulate export demand functions for Italian and EZ exports to RoW,

$$C_t^{oi} = s^{oi} \cdot \left(e_t^{io}/P_t\right)^{\sigma^r}, \quad C_t^{oe} = s^{oe} \cdot \left(e_t^{eo}/P_t^e\right)^{\sigma^r}. \tag{90}$$

Now all export demands are specified, giving

$$E_t^x = C_t^{ei} + C_t^{oi}, \quad E_t^{x,e} = C_t^{ie} + I_t^{ie} + C_t^{oe}, \quad E_t^{x,o} = C_t^{io} + I_t^{io} + C_t^{eo}. \tag{91}$$

With zero net foreign assets, trade must be balanced

$$TB_t^o = P^o E_t^{x,o} - P_t^{oe} C_t^{oe} - P_t^{oi} C_t^{oi} = 0. \tag{92}$$

Lemma 3 ***(Walras' Law)*** *In the world economy, the sum of trade balances, converting them into the same currency (e.g. Lire), must add up to zero,*

$$TB_t + e_t^{ie} TB_t^e + e_t^{io} TB_t^o = 0. \tag{93}$$

Since RoW is closed to external capital flows, global capital market clearing, expressed in the same currency, requires

$$A_t + e_t^{ie} A_t^e = B_t^f + e_t^{ie} B_t^{f,e} = 0. \tag{94}$$

Proof. Substitute the definitions of trade balances and of exports in (91) and use $P_t^{ie} = e_t^{ie} P_t^e$, $P_t^{io} = e_t^{io} P^o$ and $P_t^{ei} = P_t / e_t^{ie}$, $P_t^{oi} = P_t / e_t^{io}$, and finally $P_t^{oe} = e_t^{oe} P_t^e$, $P_t^{eo} = e_t^{eo} P^o$, $e_t^{io} e_t^{oe} = e_t^{ie}$ and $e_t^{ie} e_t^{eo} = e_t^{io}$ to get the result in (93). Imports of one country are the exports of another, and the trade surplus of one is a deficit of another.

National capital market clearing requires $A_t^j = B_t^{f,j}$. To show world market clearing in (94), substitute current accounts (71) and (86),

$$B_t^f + e_t^{ie} B_t^{f,e} = (1 + i_{t-1}) \left(B_{t-1}^f + e_{t-1}^{ie} B_{t-1}^{f,e}\right) + \left(TB_t + e_t^{ie} TB_t^e\right). \tag{i}$$

Market clearing in $t-1$ implies $B_{t-1}^f + e_{t-1}^{ie} B_{t-1}^{f,e} = 0$. By (93), $TB_t + e_t^{ie} TB_t^e = 0$ since trade with RoW is balanced, $TB_t^o = 0$, which gives global capital market clearing. ■

2.3 Currency Union

Common Monetary Policy: In a *currency union*, there is only 1 monetary policy subject to 1 money market clearing, and the exchange rate is fixed at $e^{ie} = 1$. We use weights $s_t^Y \equiv Y_t / (Y_t + Y_t^e)$ and $1 - s_t^Y$ equal to the calibrated shares in total Eurozone output. We define a 'price index' $\bar{P}_t^u$ and get

$$Y_t^u \equiv Y_t + Y_t^e, \quad \bar{P}_t^u \equiv s_t^Y \bar{P}_t + \left(1 - s_t^Y\right) \bar{P}_t^e, \quad 1 + \pi_t^u \equiv \frac{\bar{P}_{t+1}^u}{\bar{P}_t^u}, \quad s_t^Y \equiv \frac{Y_t}{Y_t^u}. \tag{95}$$

Potential output in the total Eurozone is $\bar{Y}_t^u \equiv \bar{Y}_t + \bar{Y}_t^e$, with regional levels of potential output given by (68) and (85). In a SS, $\bar{Y} = Y$ and $\bar{Y}^e = Y^e$, so that $\bar{Y}^u = Y^u$. Nominal GDP value is $Y_t^u \bar{P}_t^u = Y_t \bar{P}_t + Y_t^e \bar{P}_t^e$ since $s_t^Y Y_t^u = Y_t$ and $\left(1 - s_t^Y\right) Y_t^u = Y_t^e$.

Money market clearing requires $M^{s,u} \equiv M^s + M^{s,e} = \bar{P}\bar{M} + \bar{P}^e \bar{M}^e$. The trend component of money supply is $M^{s,u} = \phi^{m,u} \bar{Y}^u$ with a coefficient $\phi^{m,u} \equiv \left(\bar{P}\bar{M} + \bar{P}^e \bar{M}^e\right) / \bar{Y}^u$. The common monetary policy rule includes trend and countercyclical components,

$$M_t^{s,u} = (1 - \rho^m) \phi^{m,u} \bar{Y}_t^u \cdot \frac{\left(\bar{Y}_t^u / Y_t^u\right)^{\psi^y}}{(1 + \pi_t^u)^{\psi^\pi}} + \rho^m M_{t-1}^{s,u} + \varepsilon_t^{m,u}. \tag{96}$$

We allocate money supply $M_t^{s,u} \equiv M_t^s + M_t^{s,e} = \bar{P}_t \bar{M}_t + \bar{P}_t^e \bar{M}_t^e$ to accommodate money demand in each region, $M_t^s = \bar{P}_t \bar{M}_t$ and $M_t^{s,e} = \bar{P}_t^e \bar{M}_t^e$. In this case, seignorage income cancels from household budgets and Walras' Law holds separately for each country.

If starting from a SS, and if no 'other' shock occurs, a Eurozone exit cannot have any effect! To see this, note that money supply is stationary in a SS without a cyclical component, and is equal to $M^s = \phi^m \bar{Y}$ and $M^{s,e} = \phi^{m,e} \bar{Y}^e$ in the two member states. Adding up, expanding by $\bar{Y}^u = s^Y \bar{Y} + \left(1 - s^Y\right) \bar{Y}^e$ and using the definitions above gives

$$M^s + M^{s,e} = \phi^m \bar{Y} + \phi^{m,e} \bar{Y}^e = \left[\phi^m \frac{\bar{Y}}{\bar{Y}^u} + \phi^{m,e} \frac{\bar{Y}^e}{\bar{Y}^u}\right] \bar{Y}^u = \phi^{m,u} \bar{Y}^u = M^{s,u}.$$

Nothing changes if total money supply is replaced by its two components which follow exactly the same policy rules. For this to be the case, the coefficients must be related by $\phi^{m,u} = \phi^m \frac{\bar{Y}}{\bar{Y}^u} + \phi^{m,e} \frac{\bar{Y}^e}{\bar{Y}^u}$. Total money supply is the sum of the two components if and only if the coefficient $\phi^{m,u}$ is defined in this way. In addition, if the two countries are in a stationary equilibrium, there is no reason for the exchange rate $e^{ie} = 1$ to adjust, which proves the claim. We use this scenario to check consistency of our model.

Monetary Regimes: The binary variable $EZ \in \{1, 0\}$ indicates the monetary regime. Setting $EZ = 1$ implements the currency union with common monetary policy while $EZ = 0$ refers to the exit scenario with autonomous, separate policies:

$$\begin{array}{rcl|cl} & & \text{stay in EZ } (EZ=1) & & \text{exit } (EZ=0) \\ \hline M_t^s & = & EZ \cdot \bar{P}_t \bar{M}_t & + & (1-EZ) \cdot M_t^s \, (67)\,, \\ M_t^{s,e} & = & EZ \cdot \bar{P}_t^e \bar{M}_t^e & + & (1-EZ) \cdot M_t^{s,e} \, (85)\,, \\ 0 & = & EZ \cdot \left(\bar{P}_t \bar{M}_t + \bar{P}_t^e \bar{M}_t^e - M_t^{s,u} \right) & + & (1-EZ) \cdot \left(\bar{P}_t \bar{M}_t - M_t^s \right), \\ 0 & = & EZ \cdot \left(e_t^{ie} - 1 \right) & + & (1-EZ) \cdot \left(\bar{P}_t^e \bar{M}_t^e - M_t^{s,e} \right). \end{array} \tag{97}$$

In both scenarios, we compute total money supply $M_t^{s,u}$ as in (96), although it will only be used in the EMU scenario. The column ($EZ = 1$) refers to common monetary policy. The exchange rate is fixed at $e_t^{ie} = 1$, and the area wide money market clears. The first two elements in the column ($EZ = 1$) indicate that money supply flows to each region according to money demand. The second column ($EZ = 0$) refers to an exit, switching to two independent policies that separately determine M_t^s and $M_t^{s,e}$ by the rules (67) and (85). Each money market clears in isolation and the exchange rate e_t^{ie} floats freely.

3 Appendix: Wage Setting

We show that (19) exactly corresponds to Gali (2015). Note that w_t^* remains fixed until the next opportunity arrives. Shift forward (19.ii) by one period and rearrange (i),

$$(i) \quad : \quad 0 = -\left[\frac{(1-\tau_t)\, w_t^*}{(1+\tau_t^c)\, \bar{P}_t} - \frac{\sigma}{\sigma - 1} MRS_{t,t} \right] \frac{N_{t,t}}{w_t^*} + \omega \frac{\mu_{t+1}^w}{1+r_t},$$

$$(ii) \quad : \quad \mu_{t+1}^w = -\left[\frac{(1-\tau_{t+1})\, w_t^*}{\left(1+\tau_{t+1}^c\right) \bar{P}_{t+1}} - \frac{\sigma}{\sigma - 1} MRS_{t,t+1} \right] \frac{N_{t,t+1}}{w_t^*} + \omega \frac{\mu_{t+2}^w}{1+r_{t+1}}.$$

Substitute $\frac{1}{1+r_t} = \beta_t \frac{u_{\bar{C},t+1}}{u_{\bar{C},t}}$ from (15), multiply by $u_{\bar{C},t}$, and define $v_{t+i} \equiv u_{\bar{C},t+i}\mu^w_{t+i}$,

$$\begin{aligned}
(i) &: \quad 0 = -\left[\frac{(1-\tau_t)\, w_t^*}{(1+\tau_t^c)\,\bar{P}_t} - \frac{\sigma}{\sigma-1} MRS_{t,t}\right] \frac{N_{t,t}}{w_t^*} u_{\bar{C},t} + \omega\beta_t \cdot v_{t+1}, \\
(ii) &: \quad v_{t+1} = -\left[\frac{(1-\tau_{t+1})\, w_t^*}{\left(1+\tau_{t+1}^c\right)\bar{P}_{t+1}} - \frac{\sigma}{\sigma-1} MRS_{t,t+1}\right] \frac{N_{t,t+1}}{w_t^*} u_{\bar{C},t+1} + \omega\beta_{t+1} \cdot v_{t+2}.
\end{aligned}$$

Define $X_{t,t+i} \equiv \left[\frac{(1-\tau_t)w_t^*}{\left(1+\tau_{t+i}^c\right)\bar{P}_{t+i}} - \frac{\sigma}{\sigma-1} MRS_{t,t+i}\right] \frac{N_{t,t+i}}{w_t^*} u_{\bar{C},t+i}$ and note that w_t^* remains fixed. In general, the second equation is $v_{t+i} = -X_{t,t+i} + \beta_{t+i}\omega \cdot v_{t+i+1}$. Keeping β constant and solving forward establishes $0 = -\sum_{i\geq 0} (\beta\omega)^i X_{t,t+i}$. Substitute $X_{t,t+i}$ and note that w_t^* is constant (set today and unchanged thereafter) and cancels out,

$$0 = \sum_{i\geq 0} (\beta\omega)^i \left[\frac{(1-\tau_t)\, w_t^*}{\left(1+\tau_{t+i}^c\right)\bar{P}_{t+i}} - \frac{\sigma}{\sigma-1} MRS_{t,t+i}\right] N_{t,t+i} u_{\bar{C},t+i}.$$

Setting taxes to zero, this exactly corresponds to equation 11 of Gali (2015, p. 168).

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