Pension Funds as Institutions for Intertemporal Risk Transfer

R. Baumann, H. Müller, University of St. Gallen

Abstract

A continuous time overlapping generation model is used to analyse defined-contribution pension plans. Without intergenerational risk transfer between employees the optimal investment strategy results from the Merton model. Introducing intergenerational risk transfer leads to an increase in the risk tolerance of future employees and allows to improve their anticipated expected utility resulting from accrued retirement benefits. Of course, this leads to a risk of temporary underfunding. But even for an underfunded pension plan one can guarantee that in the long run, the median of the funding plan exceeds one.

Keywords: Pension finance; Defined-contribution pension fund; Intergenerational risk transfer; Overlapping generation model.

1 Introduction

Due to demographic changes and an increasing life expectancy pension finance attracts more and more interest. Among pension plans one has to distinguish between defined-benefit and defined-contribution plans. For a defined-benefit plan benefits are defined in advance and the contributions by the plan sponsor and by employees have to be adjusted. Since in the case of a defined-benefit plan the plan sponsor has to bear substantial financial risk and since it is difficult to value the accrued retirement benefits for employees changing their plan sponsor, this type of pension plan has become less popular in the recent past. Of growing importance are defined-contribution plans. A defined-contribution plan in a narrow sense invests the accrued retirement benefits of employees, the fund return is attributed to employee accounts and at retirement each employee gets the amount from his account, respectively a corresponding pension. Some defined-contribution plans (e.g. 401k plans in the US, Swedish pension funds) allow their employees to choose individual investment policies. Hence a defined-contribution plan in the narrow sense is just a broker between the employees on the one hand and financial and life insurance markets on the other hand. It bears no risk and offers a maximum of flexibility to the employees. Many researchers favour this type of pension fund. However, there are many defined-contribution plans (Continental Europe and in particular Switzerland), where the plan sponsor bears substantial risk in order to protect employees against adverse movements in financial markets. For these plans the financial situation can be represented by the funding ratio, which is the ratio of the value of assets to the present value of net obligations. Typically, for

these plans the return attributed to the individual accounts of employees depends not only on the fund return but as well on the funding ratio. To assure equal treatment, in such a plan employees cannot choose individual investment policies. However, in case of a crash on financial markets by decreasing the funding ratio the fund can protect to some extent the accrued benefits of retiring employees. In fact during 2001 - 2003 benefits of retiring employees in Continental Europe were partly protected in this way, whereas retiring members of 401k-plans suffered considerable financial losses. During prospering markets in the late nineties members of 401k-plans were better off. Thus a plan-policy which increases the funding ratio if markets perform well and decreases it in case of poor market performance leads to an intergenerational transfer of financial risk among employees. However, a pension plan of this type suffers from a lack of flexibility. A pension plan with intergenerational risk transfer for a basic pension combined with a fully flexible defined-contribution plan in the narrow sense for an extra pension could be advantageous. Employees could compensate the rigid investment strategy of the basic scheme by an appropriate investment choice in the supplementary scheme. The intergenerational risk transfer in the basic scheme would partly protect them against adverse market movements.

Asset liability portfolio problems were analyzed by several authors. Merton models with state variables (see Merton(1969)) and in particular Adler and Dumas(1983) deal with this topic in continuous time. In discrete time there are articles by Solnik(1978), Wise(1984), Wilkie(1985), Sharpe and Tint(1990), Keel and Mueller(1999), Leippold, Trojani and Vanini (2004). Further, there exists an extended and sophisticated literature on defined-contribution plans. Battocchio and Menoncin (2004) optimized the investment strategy of a defined contribution plan facing uncertainty about inflation and contributions. Boulier, Huang and Taillard (2001) and Deelstra, Graselli and Koehl (2003,2004) analyzed a defined-contribution plan providing a guarantee. However, the aim of these studies was to optimize pension wealth.

The present paper focusses on the funding ratio process. But in contrast to the articles by Browne (1999), Denzler, Müller and Scherer (2001), Müller and Baumann (2006) the liability is not exogenous. The return on employees accrued retirement benefits is linked to the funding ratio. Unlike Baumann (2005) or Deelstra, Graselli and Koehl (2003,2004), rather than optimizing the funding ratio process we concentrate on the anticipated expected utility of present and future employees. This approach allows to discuss the effects of financial intergenerational risk transfer. Such risk transfers were analyzed in a general context by Allen and Gale (1997).

In this paper we develop a simplified model for a defined-contribution pension fund which allows to analyze the impact of intergenerational risk transfer. It turns out that intergenerational risk transfer leads to an increase in risk tolerance and a Pareto improvement of ex ante expected utility for present and future employees. In section 2 a standard overlapping generation model for employees is presented. Without intergenerational risk transfer each employee has to solve a Merton Problem. In section 3 intergenerational risk transfer is introduced. Present and future employees prefer an investment strategy

by the pension plan which is more aggressive than the Merton solution in section 2 and it is shown how a Pareto improvement can be achieved. Moreover, the properties of the stochastic process representing the funding ratio is analyzed in detail. Finally, section 4 contains some general remarks and conclusions.

2 Standard Overlapping Generation Model for Employees

Employees are active for a time period τ . At each period of time $t \in \mathbb{R}$ employees are entering active life and stay there until $t + \tau$. In order to simplify the analysis we assume that employees entering at t immediately invest an amount $X_{t,0}$ ¹ which leads to accrued retirement benefits $X_{t,\tau}$ in $t + \tau$ (see figure 1).

Figure 1: Overlapping Generation Model

All employees have the same constant relative risk aversion $R > 1$ with respect to their accrued retirement benefits $X_{t,\tau}$. Hence, their utility function is given by

$$u(w) = (1 - R)^{-1} w^{1-R}.$$

There is a riskless investment opportunity $i = 0$ and risky opportunities $i = 1, \dots, N$ whose price processes are given by geometric Brownian motions, i.e.

$$\begin{aligned} \frac{dS_{0,t}}{S_{0,t}} &= r dt, \\ \left(\frac{dS_{i,t}}{S_{i,t}} \right)_{i=1,\dots,N} &= (r + \pi_i)_{i=1,\dots,N} dt + \sigma d\mathbf{Z}_t, \end{aligned} \tag{1}$$

where $\mathbf{Z}_t$ denotes the N -dimensional standard Brownian Motion. σ is a regular matrix.

In this section we assume that there is no interaction between employees being active in different time intervals. Each employee can decide how the amount $X_{t,0}$ is invested in financial markets and bears the full risk and return. According to Merton (1971) it is optimal for each employee to invest at each point of time $t + s$ his accrued retirement benefits $X_{t,s}$ in fixed proportions. Hence, the optimal investment strategy can be represented by a portfolio $\mathbf{x} \in \mathbb{R}^N$, which means that the amount $x_i \cdot X_{t,s}$ is invested in asset $i = 1, \dots, N$ and $\left(1 - \sum_{i=1}^N x_i\right) X_{t,s}$ in the riskless assets. Merton (1971) shows that the optimal portfolio is given by

$$\mathbf{x}^M = \frac{1}{R} (\sigma \sigma^\top)^{-1} \boldsymbol{\pi},$$

¹Since we are only interested in qualitative aspects we make this assumption in order to avoid complexity. Of course it would be more realistic to deal with a continuous flow of contributions from t to $t + \tau$.

where

$$\boldsymbol{\pi}^\top = (\pi_1, \dots, \pi_N).$$

Therefore, our simplified model for a defined-contribution plan in the narrow sense leads to the Merton solution and to the corresponding expected utility level for employees. We shall refer to this result in the next section, where a model for a defined-contribution plan with intergenerational risk transfer is presented.

3 Risk Transfer Model

In this section we analyze a simplified model for a defined contribution plan with intergenerational risk transfer.

3.1 Pension Fund

The intergenerational risk transfer is implemented by a fund, which is modelled as follows:

A_t : value of assets at time t

L_t : value of liabilities at time t

$F_t = \frac{A_t}{L_t}$: funding ratio at time t

The investment opportunities are described by (1) of section 2 and the fund chooses an investment policy $\mathbf{x}_t \in \mathbb{R}^N$, $t \geq 0$, in order to invest A_t . This leads to

$$dA_t = A_t(r + \mathbf{x}_t^\top \boldsymbol{\pi})dt + C_t dt + A_t \mathbf{x}_t^\top \sigma d\mathbf{Z}_t, \quad (2)$$

where C_t denotes the net contributions to the fund.

On the liability side the fund attributes a return

$$r + k \ln \left(\frac{F_t}{\overline{F}} \right), \quad k > 0, \quad (3)$$

to the accrued retirement benefit accounts of employees. The return of employees depends uniquely on the funding ratio F_t . If the funding ratio falls below the critical level $\overline{F}$, then employees get less than the riskfree rate. In this way, high and low asset returns of the fund are spread over future dates and an intergenerational risk transfer takes place.

For the liabilities L_t of the fund we get therefore

$$dL_t = L_t \left(r + k \ln \left(\frac{F_t}{\bar{F}} \right) \right) dt + C_t dt. \quad (4)$$

From

$$\ln(F_t) = \ln(A_t) - \ln(L_t)$$

one obtains

$$\begin{aligned} d(\ln(F_t)) &= d(\ln(A_t)) - d(\ln(L_t)), \\ d(\ln(F_t)) &= \frac{dA_t}{A_t} - \frac{1}{2} \mathbf{x}_t^\top \sigma \sigma^\top \mathbf{x}_t dt - \frac{dL_t}{L_t}, \\ d(\ln(F_t)) &= \left[\mathbf{x}_t^\top \boldsymbol{\pi} - k \ln \left(\frac{F_t}{\bar{F}} \right) - \frac{1}{2} \mathbf{x}_t^\top \sigma \sigma^\top \mathbf{x}_t + C_t \left(\frac{1}{A_t} - \frac{1}{L_t} \right) \right] dt + \mathbf{x}_t^\top \sigma d\mathbf{Z}_t. \end{aligned} \quad (5)$$

For further analysis we need to assume zero net contributions.

Assumption

$$C_t = 0. \quad (6)$$

Under this assumption and by substituting

$$Y_t = \ln(F_t), \quad (7)$$

one obtains the Ornstein Uhlenbeck process

$$dY_t = \left(-kY_t + k \ln \bar{F} + \mathbf{x}_t^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}_t^\top V \mathbf{x}_t \right) dt + \mathbf{x}_t^\top \sigma d\mathbf{Z}_t \quad (8)$$

with

$$V = \sigma \sigma^\top.$$

3.2 Accrued Retirement Benefits

The accrued retirement benefits $X_{t,s}$ of an employee who entered active life in t are given by

$$\frac{dX_{t,s}}{X_{t,s}} = \left(r + k \ln \left(\frac{F_{t+s}}{\bar{F}} \right) \right) ds$$

or

$$d \ln(X_{t,s}) = \left(r + k \ln \left(\frac{F_{t+s}}{\bar{F}} \right) \right) ds. \quad (9)$$

At retirement in $t + \tau$ the employee receives $X_{t,\tau}$.

3.3 Optimal Investment Policies for Different Generations

A natural objective of the fund would be to choose an investment policy which maximizes the discounted expected utility of employees averaged over all generations. To begin with we determine the investment policy of the fund which would be optimal from the point of view of an employee who enters active life at t . In other words, the expected utility of the corresponding accrued retirement benefits $X_{t,\tau}$ in $t + \tau$ has to be optimized.

Proposition 1

The investment policy $\mathbf{x}_{t'}$, $0 \leq t' \leq t + \tau$, maximizing

$$E_0 \left[(1 - R)^{-1} X_{t,\tau}^{1-R} \right]$$

is given by

$$\mathbf{x}_{t'} = \begin{cases} \frac{1}{1 + (R - 1)(1 - e^{-k\tau})e^{k(t'-t)}} V^{-1}\pi, & 0 \leq t' \leq t \\ \frac{1}{1 + (R - 1)(1 - e^{k(t'-t-\tau)})} V^{-1}\pi, & t \leq t' \leq t + \tau \end{cases} \quad (10)$$

Proof. See Appendix A1. ■

Comments

1) Figure 2 illustrates the implied risk aversion, which is given by

$$\mathcal{R}(t') = \begin{cases} 1 + (R - 1)(1 - e^{-k\tau})e^{k(t'-t)}, & 0 \leq t' \leq t \\ 1 + (R - 1)(1 - e^{k(t'-t-\tau)}), & t \leq t' \leq t + \tau \end{cases} \quad (11)$$

The investment policy becomes more conservative until the employee enters active life in t . Afterwards it becomes more aggressive and at retirement in $t + \tau$ the growth optimum portfolio is attained.

2) For all $t' \in [0, t + \tau]$ the investment policy $\mathbf{x}_{t'}$ is more aggressive than the Merton portfolio $\mathbf{x}^M$ in section 2.

3.4 Static Investment Policies

The complexity arising from the aggregation of discounted expected utilities over all generations does not allow to get an explicit solution for the corresponding dynamic investment policy. Therefore, we analyze static investment policies from now on.

A static investment policy is given by the choice of a portfolio $\mathbf{x} \in \mathbb{R}^N$. Then for $Y_t = \ln(F_t)$ one obtains the Ornstein Uhlenbeck process

$$dY_t = \left(-kY_t + k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) dt + \mathbf{x}^\top \sigma d\mathbf{Z}_t. \quad (12)$$

Figure 2: Implied Risk Aversion

As shown in Appendix A2, the solution is

$$Y_t = e^{-kt} Y_0 + \left(\ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) (1 - e^{-kt}) + \int_0^t e^{k(s-t)} \mathbf{x}^\top \sigma d\mathbf{Z}_s. \quad (13)$$

If Y_0 is Gaussian or deterministic then Y_t is Gaussian with

$$E(Y_t) = e^{-kt} E(Y_0) + \left(\ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) (1 - e^{-kt}), \quad (14)$$

$$\begin{aligned} Var(Y_t) &= e^{-2kt} Var(Y_0) + \int_0^t e^{2k(s-t)} \mathbf{x}^\top V \mathbf{x} ds \\ &= e^{-2kt} Var(Y_0) + \frac{\mathbf{x}^\top V \mathbf{x}}{2k} (1 - e^{-2kt}). \end{aligned} \quad (15)$$

From (13), (14), (15) one observes immediately that

$$Y_0 = 0, \quad \ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} > 0$$

implies $med(F_t) > 1$ for all t .

Low values of k lead to high volatility of F_t . Later on, we shall discuss the funding ratio F_t in more detail.

In Appendix A3 the following formula for the accrued retirement benefits $X_{t,\tau}$ is derived:

$$\begin{aligned} \ln(X_{t,\tau}) &= \ln(X_{t,0}) + \left(r + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) \tau + (1 - e^{-k\tau}) \ln(F_t) \\ &\quad - \left(\ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) (1 - e^{-k\tau}) + \int_0^\tau (1 - e^{k(s-t)}) \mathbf{x}^\top \sigma d\mathbf{Z}_{t+s}. \end{aligned} \quad (16)$$

The term $\ln(X_{t,\tau})$ is Gaussian and its first two conditional moments with respect to F_0 are given by

$$\begin{aligned}
E_0 [\ln(X_{t,\tau} | F_0)] &= \ln(X_{t,0}) + \left(r + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) \tau \\
&\quad + (1 - e^{-k\tau}) \left[e^{-kt} \ln(F_0) + \left(\ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) (1 - e^{-kt}) \right] \\
&\quad - \left(\ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) (1 - e^{-k\tau}), \\
E_0 [\ln(X_{t,\tau} | F_0)] &= \ln(X_{t,0}) + \left(r + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) \tau \\
&\quad + (1 - e^{-k\tau}) e^{-kt} \ln(F_0) - e^{-kt} (1 - e^{-k\tau}) \left(\ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right), \quad (17)
\end{aligned}$$

$$\begin{aligned}
Var_0 [\ln(X_{t,\tau} | F_0)] &= \mathbf{x}^\top V \mathbf{x} \left[\tau - \frac{2}{k} (1 - e^{-k\tau}) + \frac{1}{2k} (1 - e^{-2k\tau}) \right] + \frac{\mathbf{x}^\top V \mathbf{x}}{2k} (1 - e^{-2kt}) (1 - e^{-k\tau})^2, \\
Var_0 [\ln(X_{t,\tau} | F_0)] &= \mathbf{x}^\top V \mathbf{x} \left\{ \tau - \frac{1 - e^{-k\tau}}{k} \left[1 + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right] \right\}. \quad (18)
\end{aligned}$$

3.5 Optimal Static Investment Strategies

Before aggregating the preferences of employees we look for the most preferred static strategy from the point of view of an employee entering active life at t .

Applying the utility function of section 2 on the accrued retirement benefits $X_{t,\tau}$ leads to

$$\begin{aligned}
u(X_{t,\tau}) &= (1 - R)^{-1} (X_{t,\tau})^{1-R} \\
&= (1 - R)^{-1} e^{(1-R) \ln X_{t,\tau}}
\end{aligned}$$

Since $\ln(X_{t,\tau})$ is Gaussian, the anticipated expected utility at time 0 is given by

$$E_0 [u(X_{t,\tau} | F_0)] = (1 - R)^{-1} \exp \left\{ (1 - R) E_0 [\ln(X_{t,\tau} | F_0)] + \frac{1}{2} (1 - R)^2 Var_0 [\ln(X_{t,\tau} | F_0)] \right\}. \quad (19)$$

The investment policy of the fund is determined by the choice of a portfolio $\mathbf{x} \in \mathbb{R}^N$. For an employee entering active life at time t the choice of a portfolio $\mathbf{x}(t)$, which maximizes his anticipated expected utility would be optimal. Obviously, $\mathbf{x}(t)$ results from

$$\max_{\mathbf{x} \in \mathbb{R}^N} \left\{ E_0 [\ln(X_{t,\tau} | F_0)] + \frac{1}{2} (1 - R) Var_0 [\ln(X_{t,\tau} | F_0)] \right\}. \quad (20)$$

According to (17), (18) the optimization problem can be reduced to

$$\max_{\mathbf{x} \in \mathbb{R}^N} \left\{ \left(\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) \left[\tau - \frac{1 - e^{-k\tau}}{k} e^{-kt} \right] + \frac{1}{2} (1 - R) \mathbf{x}^\top V \mathbf{x} \left\{ \tau - \frac{1 - e^{-k\tau}}{k} \left[1 + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right] \right\} \right\}$$

or

$$\max_{\mathbf{x} \in \mathbb{R}^N} \left\{ \mathbf{x}^\top \boldsymbol{\pi} \left[\tau - \frac{1 - e^{-k\tau}}{k} e^{-kt} \right] - \frac{R}{2} \mathbf{x}^\top V \mathbf{x} \left\{ \tau - \frac{1 - e^{-k\tau}}{k} \left[1 + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right] + \frac{1}{R} \frac{1 - e^{-k\tau}}{k} \left[1 - e^{-kt} + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right] \right\} \right\}. \quad (21)$$

Proposition 2

1) The optimal portfolio for an employee entering active life at t is

$$\mathbf{x}(t) = \frac{1}{R} \frac{c(t, k, \tau)}{d(t, k, \tau)} V^{-1} \boldsymbol{\pi}, \quad (22)$$

where

$$c(t, k, \tau) = \tau - \frac{1 - e^{-k\tau}}{k} e^{-kt},$$

$$d(t, k, \tau) = \tau - \frac{1 - e^{-k\tau}}{k} \left[1 + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right] + \frac{1}{R} \frac{1 - e^{-k\tau}}{k} \left[1 - e^{-kt} + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right].$$

2) $\frac{1}{R} < \frac{c(t, k, \tau)}{Rd(t, k, \tau)} < 1$, $d(t, k, \tau) > 0$ for all t, k, τ .

Proof. See Appendix A4. ■

Comments

- 1) Thus, in the risk transfer model all employees prefer portfolios $\mathbf{x}(t)$ which are more aggressive than the Merton solution $\mathbf{x}^M$ of section 2, but less aggressive than the growth optimum portfolio.
- 2) $\mathbf{x}(t)$ does not depend on F_0, F_t or $\overline{F}$.

3.6 Risk Tolerance for Different Generations of Employees

The risk tolerance of an employee entering active life at time t is given by

$$\theta(t) = \frac{1}{R} \frac{c(t, k, \tau)}{d(t, k, \tau)}.$$

We already know that $1 > \theta(t) > \frac{1}{R}$ holds for all $t \geq 0$.

The next proposition contains more information about $\theta(t)$.

Proposition 3

- 1) $\theta(0) < \lim_{t \rightarrow \infty} \theta(t) < 1$
- 2) There exists $T^* > 0$ not depending on R such that

$\theta(t)$ is strictly decreasing for $0 < t < T^*$.

$\theta(t)$ is strictly increasing for $t > T^*$.

Proof. See Appendix A5. ■

Figure 3 illustrates the shape of $\theta(t)$.

Comments

Figure 3: Shape of Risk Tolerance

- 1) The shape of $\theta(t)$ is not surprising. Employees entering active life in the near future suffer from the risk with respect to the future funding ratio. For employees entering in the far distant future, expected return plays a more important role.
- 2) According to (21) the optimal portfolio $\mathbf{x}(t)$ for an employee entering active life at time t results from

$$\max_{\mathbf{x} \in \mathbb{R}^N} \left(\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2\theta(t)} \mathbf{x}^\top V \mathbf{x} \right),$$

and we get

$$\mathbf{x}(t) = \theta(t) V^{-1} \boldsymbol{\pi}.$$

Therefore, all portfolio choices

$$\mathbf{x} = \lambda V^{-1} \boldsymbol{\pi},$$

$$\theta(T^*) \leq \lambda < \lim_{t \rightarrow \infty} \theta(t),$$

are Pareto efficient.

3.7 Welfare Optimizing Static Investment Strategies

The objective function of an employee entering active life at t is given by (20). Therefore we introduce the welfare function

$$\int_0^\infty e^{-\delta t} \left\{ E_0 \left[\ln \left(\frac{X_{t,\tau}}{X_{t,0}} \right) \right] + \frac{1-R}{2} Var_0 \left[\ln \left(\frac{X_{t,\tau}}{X_{t,0}} \right) \right] \right\} dt. \quad (23)$$

According to (20), (21) this leads to the objective function

$$H(\mathbf{x}) = \int_0^\infty \left\{ \mathbf{x}^\top \boldsymbol{\pi} \left(\tau e^{-\delta t} - \frac{1-e^{-k\tau}}{k} e^{-(k+\delta)t} \right) - \frac{R}{2} \mathbf{x}^\top V \mathbf{x} \left\{ \begin{aligned} &\tau e^{-\delta t} - \frac{1-e^{-k\tau}}{k} \left[e^{-\delta t} + \frac{1}{2} e^{-(2k+\delta)t} (1 - e^{-k\tau}) \right] \\ &+ \frac{1}{R} \frac{1-e^{-k\tau}}{k} \left[e^{-\delta t} - e^{-(k+\delta)t} + \frac{1}{2} e^{-(2k+\delta)t} (1 - e^{-k\tau}) \right] \end{aligned} \right\} \right\} dt$$

or

$$\begin{aligned} H(\mathbf{x}) &= \mathbf{x}^\top \boldsymbol{\pi} \left(\frac{\tau}{\delta} - \frac{1-e^{-k\tau}}{k} \frac{1}{k+\delta} \right) \\ &\quad - \frac{R}{2} \mathbf{x}^\top V \mathbf{x} \left\{ \begin{aligned} &\frac{\tau}{\delta} - \frac{1-e^{-k\tau}}{k} \left[\frac{1}{\delta} + \frac{1}{2} \frac{1}{2k+\delta} (1 - e^{-k\tau}) \right] \\ &+ \frac{1}{R} \frac{1-e^{-k\tau}}{k} \left[\frac{1}{\delta} - \frac{1}{k+\delta} + \frac{1}{2} \frac{1}{2k+\delta} (1 - e^{-k\tau}) \right] \end{aligned} \right\}. \end{aligned} \quad (24)$$

Proposition 4

1) The welfare maximizing portfolio is given by

$$\mathbf{x} = \frac{1}{R} \frac{c(\delta, k, \tau)}{d(\delta, k, \tau)} V^{-1} \boldsymbol{\pi}, \quad (25)$$

where

$$\begin{aligned} c(\delta, k, \tau) &= \frac{\tau}{\delta} - \frac{1-e^{-k\tau}}{k} \frac{1}{k+\delta}, \\ d(\delta, k, \tau) &= \frac{\tau}{\delta} - \frac{1-e^{-k\tau}}{k} \left\{ \left(1 - \frac{1}{R} \right) \left[\frac{1}{\delta} + \frac{1}{2} \frac{1}{2k+\delta} (1 - e^{-k\tau}) \right] + \frac{1}{R} \frac{1}{k+\delta} \right\}. \end{aligned}$$

2) $\frac{1}{R} < \frac{c(\delta, k, \tau)}{Rd(\delta, k, \tau)} < 1$, $d(\delta, k, \tau) > 0$ for all δ, k, τ .

Proof. See Appendix A6. ■

Comments

- 1) For all discount rates $\delta > 0$ the welfare maximizing portfolio is more aggressive than the Merton portfolio $\mathbf{x}^M$ of section 2, but less aggressive than the growth optimum portfolio.
- 2) Due to proposition 3 (shape of implied risk tolerance for different generations of employees) there may exist no $\delta > 0$ such that the welfare maximizing portfolio leads to a Pareto improvement relative to the Merton solution $\mathbf{x}^M$ of section 2. For every $\delta > 0$ the employee entering active life at T^* may prefer the Merton solution $\mathbf{x}^M$ to the welfare maximizing portfolio.

The next section deals with intergenerational risk transfers leading to a Pareto improvement.

3.8 Pareto Improvement due to Intergenerational Risk Transfer

Now we address the question whether the model with intergenerational risk transfer can be advantageous for all employees. Or more precisely, does there exist a portfolio $\mathbf{x}$ for the fund strategy which leads for all employees to a higher anticipated expected utility than the model in section 2.

In order to compare anticipated expected utilities, according to (19) it is sufficient to look at

$$\begin{aligned}
& E_0 [\ln(X_{t,\tau}) \mid F_0] + \frac{1}{2} (1 - R) \text{Var}_0 [\ln(X_{t,\tau}) \mid F_0] \\
= & \ln(X_{t,0}) + \left(r + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) \tau \\
& + (1 - e^{-k\tau}) e^{-kt} \ln(F_0) - e^{-kt} (1 - e^{-k\tau}) \left(\ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) \\
& + \frac{1}{2} (1 - R) \mathbf{x}^\top V \mathbf{x} \left\{ \tau - \frac{1 - e^{-k\tau}}{k} \left[1 + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right] \right\}. \tag{26}
\end{aligned}$$

Proposition 5

Assume $F_0 = 1$. Then for $\mathbf{x}^* = \lambda^* V^{-1} \boldsymbol{\pi}$ with $\lambda^* = \min_{t \geq 0} \frac{c(t,k,\tau)}{d(t,k,\tau)} \frac{1}{R}$, there exists $\bar{F} < 1$ such that

1) In comparison with the standard overlapping generation model of section 2 all employees attain a higher anticipated expected utility.

2) $\text{med}(F_t) > 1$ for all $t > 0$.

Proof. Let

$$\mathbf{x}^M = \frac{1}{R} V^{-1} \boldsymbol{\pi}$$

denote the Merton solution of section 2. Choose $\bar{F}$ such that

$$\ln(\bar{F}) + \frac{(\mathbf{x}^M)^\top \boldsymbol{\pi} - \frac{1}{2} (\mathbf{x}^M)^\top V \mathbf{x}^M}{k} = 0 \tag{27}$$

holds. From

$$\frac{1}{R} < \lambda^* < 1$$

one concludes

$$(\mathbf{x}^*)^\top \boldsymbol{\pi} - \frac{1}{2} (\mathbf{x}^*)^\top V \mathbf{x}^* > (\mathbf{x}^M)^\top \boldsymbol{\pi} - \frac{1}{2} (\mathbf{x}^M)^\top V \mathbf{x}^M.$$

This leads to

$$\ln(\bar{F}) + \frac{(\mathbf{x}^*)^\top \boldsymbol{\pi} - \frac{1}{2} (\mathbf{x}^*)^\top V \mathbf{x}^*}{k} > 0 \tag{28}$$

and we get $\text{med}(F_t) > 1$ for all $t > 0$. Taking into account (26), (27), one obtains for $\mathbf{x}^*$ and $\overline{F}$

$$\begin{aligned}
& E_0 [\ln(X_{t,\tau} \mid F_0)] + \frac{1}{2} (1-R) \text{Var}_0 [\ln(X_{t,\tau} \mid F_0)] \\
&= \ln(X_{t,0}) + \left[r + (\mathbf{x}^*)^\top \boldsymbol{\pi} - \frac{1}{2} (\mathbf{x}^*)^\top V \mathbf{x}^* \right] \tau - e^{-kt} (1 - e^{-k\tau}) \left[\ln(\overline{F}) + \frac{(\mathbf{x}^*)^\top \boldsymbol{\pi} - \frac{1}{2} (\mathbf{x}^*)^\top V \mathbf{x}^*}{k} \right] \\
&\quad + \frac{1}{2} (1-R) (\mathbf{x}^*)^\top V \mathbf{x}^* \left\{ \tau - \frac{1 - e^{k\tau}}{k} \left[1 + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right] \right\} \\
&> \ln(X_{t,0}) + \left[r + (\mathbf{x}^M)^\top \boldsymbol{\pi} - \frac{1}{2} (\mathbf{x}^M)^\top V \mathbf{x}^M \right] \tau \\
&\quad + \frac{1}{2} (1-R) (\mathbf{x}^M)^\top V \mathbf{x}^M \left\{ \tau - \frac{1 - e^{k\tau}}{k} \left[1 + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right] \right\} \\
&> \ln(X_{t,0}) + \left[r + (\mathbf{x}^M)^\top \boldsymbol{\pi} - \frac{1}{2} (\mathbf{x}^M)^\top V \mathbf{x}^M \right] \tau + \frac{1}{2} (1-R) (\mathbf{x}^M)^\top V \mathbf{x}^M \tau. \tag{29}
\end{aligned}$$

■

Comments

- 1) It is quite natural to assume that the plan starts fully funded at $t = 0$, i.e.. $F_0 = 1$.
- 2) The property $\text{med}(F_t) > 1$ is a condition for the financial stability of the pension plan. This issue will be discussed later on in more detail.
- 3) Under the portfolio $\mathbf{x}^*$ and $\overline{F} < 1$ all employees are better off than with the standard overlapping generation model of section 2. By taking advantage of the increase in risk tolerance a Pareto improvement can be achieved under the stability condition $\text{med}(F_t) > 1$.

3.9 Analysis of the Funding Ratio

For the analysis of the funding ratio F_t we assume $F_0 = 1$. Then, according to (13), $Y_t = \ln(F_t)$ is Gaussian and is given by

$$Y_t = \left(\ln(\overline{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) (1 - e^{-kt}) + \int_0^t e^{k(s-t)} \mathbf{x}^\top \sigma d\mathbf{Z}_s, \tag{13'}$$

$$E(Y_t) = \left(\ln(\overline{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) (1 - e^{-kt}), \tag{14'}$$

$$\text{Var}(Y_t) = \sigma^2(Y_t) = \frac{\mathbf{x}^\top V \mathbf{x}}{2k} (1 - e^{-2kt}). \tag{15'}$$

In the extreme case $\overline{F} = 1$, $k \rightarrow \infty$ one obtains

$$E(Y_t) = 0,$$

$$\text{Var}(Y_t) = 0,$$

and according to (22) the optimal portfolios tend towards the Merton solution

$$\mathbf{x}^M = \frac{1}{R} V^{-1} \boldsymbol{\pi}.$$

Hence, we are back in the model of section 2, where no risk transfer takes place.

For $0 < k < \infty$ we study the α -percentiles of $Y_t = \ln(F_t)$.

$$p(t, \alpha) = E(Y_t) + z_\alpha \sigma(Y_t),$$

with

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_\alpha} e^{-\frac{x^2}{2}} dx = \alpha,$$

$$p(t, \alpha) = \left(\ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) (1 - e^{-kt}) + z_\alpha \left(\frac{\mathbf{x}^\top V \mathbf{x}}{2k} \right)^{\frac{1}{2}} (1 - e^{-2kt})^{\frac{1}{2}}.$$

From now on we assume

$$\ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} > 0.$$

Then, according to (14') $\text{med}(F_t) > 1$ holds.

In the next proposition the properties of the percentiles of $Y_t = \ln(F_t)$ are summarized.

Proposition 6

- 1) $p(0, \alpha) = 0$
- 2) $\lim_{t \rightarrow \infty} p(t, \alpha) = \ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} + z_\alpha \left(\frac{\mathbf{x}^\top V \mathbf{x}}{2k} \right)^{\frac{1}{2}}$
- 3) $p(t, \alpha)$ is strictly increasing and concave in t for $\alpha \geq \frac{1}{2}$.
- 4) If $\alpha < \frac{1}{2}$, then $p(t, \alpha)$ attains a minimum at

$$t^*(\alpha) = \frac{1}{2k} \ln \left(1 + \frac{z_\alpha^2 \mathbf{x}^\top V \mathbf{x}}{2k \left(\ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right)^2} \right)$$

and

$p(t, \alpha)$ is strictly decreasing for $t \in (0, t^*(\alpha))$,

$p(t, \alpha)$ is strictly increasing for $t \in (t^*(\alpha), +\infty)$.

Proof. See Appendix A7. ■

Figure 4 illustrates the shape of the percentiles of $Y_t = \ln(F_t)$.

Comments

- 1) The shape of the percentiles provides some intuition for proposition 3. Employees entering active life in the very near or in the far distant future suffer less under the uncertainty with respect to the funding ratio than those entering in the intermediate future.

Figure 4: Percentiles of $Y_t = \ln F_t$.

- 2) In order to take substantial advantages of the intergenerational risk transfer k should not be too large. However, according to proposition 6, a small k leads to a considerable risk of underfunding., which may become disruptive for the stability of the pension plan. In other words, there is a trade-off between the advantages of intergenerational risk transfer and the risk of temporary underfunding. However, in the long run the conditional median of the funding ratio F_T always tends to a value larger than 1. This follows from (14). In fact one obtains

$$\lim_{T \rightarrow \infty} \text{med}[F_T | F_t] = \lim_{T \rightarrow \infty} e^{E[Y_T | Y_t]} = \exp \left\{ \ln(\bar{F}) + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right\} > 1.$$

4 Conclusions

In this paper we analyzed a simplified version of a defined-contribution pension plan with intergenerational risk transfer. The intergenerational risk transfer leads to an increase in implicit risk tolerance for all future employees. In fact, all future employees can attain a higher anticipated expected utility than in a defined-contribution plan without any risk transfer. In this sense a Pareto improvement can be achieved. However, such a framework is inappropriate for an individual choice of the investment policy. Moreover, despite the fact that the conditional median of the funding ratio for the far distant future exceeds hundred percents at any time, there is a risk of underfunding. Of course at the starting point of the plan intergenerational risk transfer leads to an increase of the anticipated expected utility for all future members. However, later on due to adverse market movements the plan may get underfunded and unattractive for new members. Therefore, pension plans with intergenerational risk transfer are only appropriate for perennial institutions, such as public employees, large firms, etc.

Moreover, a pension plan with intergenerational risk transfer should be supplemented by a plan with

individual investment decisions and no transfer of financial risks. This would allow to take advantage of intergenerational risk transfer for a basic pension scheme and to restore individual flexibility by means of a supplementary plan.

Appendix A.1

Proof of proposition 1.

a) Optimal investment strategy $\mathbf{x}_{t+s}$, $0 \leq s \leq t$.

According to (8), (9) the accrued retirement benefits $X_{t,s}$ and the funding ratio F_{t+s} are given by

$$dX_{t,s} = X_{t,s} \left(r + k \ln \left(\frac{F_{t+s}}{\bar{F}} \right) \right) ds, \quad 0 \leq s \leq \tau,$$

$$dF_{t+s} = F_{t+s} \left[\mathbf{x}_{t+s}^\top \boldsymbol{\pi} - k \ln \left(\frac{F_{t+s}}{\bar{F}} \right) \right] dt + F_{t+s} \mathbf{x}_{t+s}^\top \sigma d\mathbf{Z}_t.$$

This leads to the HJB-equation

$$0 = J_s + \max_{\mathbf{x}} \left\{ J_X X_{t,s} \left[r + k \ln \left(\frac{F_{t+s}}{\bar{F}} \right) \right] + J_F F_{t+s} \left[\mathbf{x}_{t+s}^\top \boldsymbol{\pi} - k \ln \left(\frac{F_{t+s}}{\bar{F}} \right) \right] + \frac{1}{2} J_{FF} F_{t+s}^2 \mathbf{x}_{t+s}^\top V \mathbf{x}_{t+s} \right\}$$

with

$$J(t + \tau, X_{t+\tau}, F_{t+\tau}) = (1 - R)^{-1} X_{t,\tau}^{1-R}.$$

The optimal investment strategy is given by

$$\mathbf{x}_{t+s} = -\frac{J_F}{F J_{FF}} V^{-1} \boldsymbol{\pi}.$$

Inserting into the HJB-equation leads to

$$0 = J_s + J_X X \left[r + k \ln \left(\frac{F}{\bar{F}} \right) \right] - J_F F k \ln \left(\frac{F}{\bar{F}} \right) - \frac{1}{2} \frac{J_F^2}{J_{FF}} \boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}.$$

For the solution one tries

$$J(t + s, X, F) = -e^{f(t+s)} X^{g(t+s)} F^{h(t+s)}.$$

Using

$$\begin{aligned} J_X X &= g(t+s) J, \\ J_F F &= h(t+s) J, \\ \frac{J_F^2}{J_{FF}} &= \frac{h(t+s)}{h(t+s) - 1}, \\ J_s &= (f'(t+s) + g'(t+s) \ln X + h'(t+s) \ln F) J. \end{aligned}$$

one obtains

$$\begin{aligned} 0 &= f'(t+s) + g'(t+s) \ln X + h'(t+s) \ln F + g(t+s) \left[r + k \ln \left(\frac{F}{\bar{F}} \right) \right] \\ &\quad - h(t+s) k \ln \left(\frac{F}{\bar{F}} \right) - \frac{1}{2} \frac{h(t+s)}{h(t+s) - 1} \boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}. \end{aligned}$$

First, from

$$\begin{aligned} g'(t+s) &= 0, \\ J(t + \tau, X, F) &= (1 - R)^{-1} X^{1-R} \end{aligned}$$

one concludes

$$g(t+s) = 1 - R.$$

Moreover, the solution of

$$\begin{aligned} h'(t+s) + (1-R)k - h(t+s)k &= 0, \\ h(t+\tau) &= 0 \end{aligned}$$

is given by

$$h(t+s) = (1-R) \left(1 - e^{k(s-\tau)}\right), \quad 0 \leq s \leq \tau.$$

Hence one obtains for the HJB-equation

$$\begin{aligned} 0 &= f'(t+s) + (1-R) \left[r - k \ln \bar{F}\right] + (1-R) \left(1 - e^{k(s-\tau)}\right) k \ln \bar{F} \\ &\quad - \frac{\boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}}{2} \frac{(1-R) \left(1 - e^{k(s-\tau)}\right)}{(1-R) \left(1 - e^{k(s-\tau)}\right) - 1}. \end{aligned}$$

The solution $f(t+s)$ is not needed later on.

Therefore, we get

$$\begin{aligned} J(t+s, X, F) &= e^{f(t+s)} X^{1-R} F^{(1-R)(1-e^{k(s-\tau)})}, \\ \mathbf{x}_{t+s} &= \frac{1}{1 + (R-1) \left(1 - e^{k(s-\tau)}\right)} V^{-1} \boldsymbol{\pi}. \end{aligned}$$

b) Optimal investment strategy for $\mathbf{x}_s$, $0 \leq s \leq t$.

For $0 \leq s \leq t$ one obtains the HJB-equation

$$0 = J_s + \max_{\mathbf{x}} \left\{ J_F F_s \left[\mathbf{x}_s^\top \boldsymbol{\pi} - k \ln \left(\frac{F_s}{\bar{F}} \right) \right] + \frac{1}{2} J_{FF} F_s^2 \mathbf{x}_s^\top V \mathbf{x}_s \right\},$$

with

$$J(t, F_t) = e^{-f(t)} X_{t,0}^{1-R} F_t^{(1-R)(1-e^{-k\tau})}.$$

The optimal investment strategy is given by

$$\mathbf{x}_s = -\frac{J_F}{F_s J_{FF}} V^{-1} \boldsymbol{\pi}.$$

Inserting into the HJB-equation leads to

$$0 = J_s - J_F F k \ln \left(\frac{F}{\bar{F}} \right) - \frac{1}{2} \frac{J_F^2}{J_{FF}} \boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}.$$

For the solution one tries

$$J(s, F) = -e^{a(s)} F^{b(s)}.$$

Using

$$\begin{aligned} J_F F &= b(s) J \\ \frac{J_F^2}{J_{FF}} &= \frac{b(s)}{b(s) - 1} \\ J_s &= (a'(s) + b'(s) \ln F) J \end{aligned}$$

one obtains

$$a'(s) + b'(s) \ln F - b(s) k (\ln F - \ln \bar{F}) - \frac{1}{2} \frac{b(s)}{b(s) - 1} \boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi} = 0.$$

From

$$\begin{aligned} b'(s) - kb(s) &= 0 \\ b(t) &= (1 - R) (1 - e^{-k\tau}) \end{aligned}$$

one concludes

$$b(s) = (1 - R) (1 - e^{-k\tau}) e^{k(s-t)}, \quad 0 \leq s \leq t.$$

This leads to

$$J(s, F) = e^{a(s)} F^{(1-R)} (1 - e^{-k\tau}) e^{k(s-t)}$$

and

$$\mathbf{x}_s = \frac{1}{1 + (R - 1) (1 - e^{-k\tau}) e^{k(s-t)}} V^{-1} \boldsymbol{\pi}, \quad 0 \leq s \leq t.$$

■

Appendix A.2

Solution of the Ornstein-Uhlenbeck process

$$dY_t = \left(-kY_t + k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) dt + \mathbf{x}^\top \sigma d\mathbf{Z}_t.$$

Using the formula in Karatzas/Shreve (1997), p. 354 with

$$\begin{aligned} A(t) &= -k \\ a(t) &= k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \\ \sigma(t) &= (\mathbf{x}^\top V \mathbf{x})^{0.5} \end{aligned}$$

one obtains $\Phi(t) = e^{-kt}$ and

$$Y_t = \Phi(t) \left[Y_0 + \int_0^t [\Phi(s)]^{-1} a(s) ds + \int_0^t [\Phi(s)]^{-1} \sigma(s) dZ_s^* \right]$$

where Z_t^* is a 1-dimensional standard Brownian motion. We get

$$\begin{aligned} Y_t &= e^{-kt} \left[Y_0 + \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) \int_0^t e^{ks} ds + (\mathbf{x}^\top V \mathbf{x})^{0.5} \int_0^t e^{ks} dZ_s^* \right] \\ &= e^{-kt} Y_0 + \left(\ln \bar{F} + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) (1 - e^{-kt}) + (\mathbf{x}^\top V \mathbf{x})^{0.5} \int_0^t e^{k(s-t)} dZ_s^* \\ &= e^{-kt} Y_0 + \left(\ln \bar{F} + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) (1 - e^{-kt}) + \int_0^t e^{k(s-t)} \mathbf{x}^\top \sigma d\mathbf{Z}_s \end{aligned}$$

which is the solution of the Ornstein-Uhlenbeck process.

Appendix A.3

Solution of

$$d \ln X_{t,s} = \left(r + k \ln \left(\frac{F_{t+s}}{\bar{F}} \right) \right) ds .$$

First we calculate

$$\begin{aligned} \int_0^\tau k \ln F_{t+s} ds &= \int_0^\tau k Y_{t+s} ds \\ &= (1 - e^{-k\tau}) Y_t + \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) \left[\tau - \frac{1}{k} (1 - e^{-k\tau}) \right] \\ &\quad + (\mathbf{x}^\top V \mathbf{x})^{0.5} \int_0^\tau \int_0^s k e^{k(u-s)} dZ_{t+u}^* ds \\ &= (1 - e^{-k\tau}) Y_t + \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) \left[\tau - \frac{1}{k} (1 - e^{-k\tau}) \right] \\ &\quad + (\mathbf{x}^\top V \mathbf{x})^{0.5} \int_0^\tau \int_u^\tau k e^{k(u-s)} ds dZ_{t+u}^* \\ &= (1 - e^{-k\tau}) Y_t + \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) \left[\tau - \frac{1}{k} (1 - e^{-k\tau}) \right] \\ &\quad + (\mathbf{x}^\top V \mathbf{x})^{0.5} \int_0^\tau (1 - e^{k(u-\tau)}) dZ_{t+u}^* . \end{aligned}$$

Since

$$(\mathbf{x}^\top V \mathbf{x})^{0.5} dZ_{t+u}^* = \mathbf{x}^\top \sigma d\mathbf{Z}_{t+u}$$

one obtains

$$\begin{aligned} \int_0^\tau k \ln F_{t+s} ds &= (1 - e^{-k\tau}) Y_t + \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) \left[\tau - \frac{1}{k} (1 - e^{-k\tau}) \right] \\ &\quad + \int_0^\tau (1 - e^{k(u-\tau)}) \mathbf{x}^\top \sigma d\mathbf{Z}_{t+u} . \end{aligned}$$

Hence integration of $d \ln X_{t,\tau}$ leads to

$$\begin{aligned} \ln X_{t,\tau} &= \ln X_{t,0} + \left(r + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) \tau \\ &\quad + (1 - e^{-k\tau}) \ln F_t + \left(\ln \bar{F} + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) (1 - e^{-k\tau}) \\ &\quad + \int_0^\tau (1 - e^{k(u-\tau)}) \mathbf{x}^\top \sigma d\mathbf{Z}_{t+u} . \end{aligned}$$

Appendix A.4

Proof of proposition 2.

1) The formula for the optimal portfolio $\mathbf{x}^*(t)$ follows directly from the optimality condition

$$c(t, k, \tau) \boldsymbol{\pi} - d(t, k, \tau) R V \mathbf{x}^* = \mathbf{0} .$$

2) Next, we show that $d(t, k, \tau) > 0$.

a)

$$\lim_{k \rightarrow 0} d(t, k, \tau) > 0$$

b)

$$kd(0, k, \tau) = k\tau - (1 - e^{-k\tau}) - \frac{1}{2} (1 - e^{-k\tau})^2 + \frac{1}{2R} (1 - e^{-k\tau})^2$$

$$\begin{aligned} \frac{\partial}{\partial k} [kd(0, k, \tau)] &= \tau - \tau(1 - e^{-k\tau}) - \tau(1 - e^{-k\tau})e^{-k\tau} + \frac{\tau}{R} (1 - e^{-k\tau})e^{-k\tau} \\ &= \tau \left[e^{-k\tau} \left(1 - \left(1 - \frac{1}{R} \right) (1 - e^{-k\tau}) \right) \right] > 0 \end{aligned}$$

for $\tau > 0, k > 0$. Taking into account a) this implies $d(0, k, \tau) > 0$ for $\tau > 0, k > 0$.

c)

$$\begin{aligned} \frac{\partial}{\partial t} [d(t, k, \tau)] &= (1 - e^{-k\tau})^2 e^{-2kt} + \frac{1}{R} (1 - e^{-k\tau}) e^{-kt} - \frac{1}{R} (1 - e^{-k\tau})^2 e^{-2kt} \\ &= \left(1 - \frac{1}{R} \right) (1 - e^{-k\tau})^2 e^{-2kt} + \frac{1}{R} (1 - e^{-k\tau}) e^{-kt} > 0. \end{aligned}$$

Hence, we have shown that

$$d(t, k, \tau) > 0 \text{ for } \tau > 0, k > 0.$$

3) Next, we show $c(t, k, \tau) > d(t, k, \tau)$. One obtains

$$\begin{aligned} c(t, k, \tau) - d(t, k, \tau) &= \tau - \frac{1 - e^{-k\tau}}{k} e^{-kt} - \tau + \frac{1 - e^{-k\tau}}{k} \left[1 + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right] \\ &\quad - \frac{1}{R} \frac{1 - e^{-k\tau}}{k} \left[1 - e^{-kt} + \frac{e^{-2kt}}{2} (1 - e^{-k\tau}) \right] \\ &= \frac{1 - e^{-k\tau}}{k} \left(1 - e^{-kt} + \frac{e^{-2kt}}{2} (1 - e^{-k\tau}) - \frac{1}{R} \left[1 - e^{-kt} + \frac{e^{-2kt}}{2} (1 - e^{-k\tau}) \right] \right) \\ &= \underbrace{\frac{1 - e^{-k\tau}}{k}}_{>0} \underbrace{\left(1 - \frac{1}{R} \right)}_{>0} \left(\underbrace{1 - e^{-kt}}_{>0} + \underbrace{\frac{e^{-2kt}}{2}}_{>0} \underbrace{(1 - e^{-k\tau})}_{>0} \right) > 0. \end{aligned}$$

4) Finally, it remains to show that

$$\frac{c(t, k, \tau)}{Rd(t, k, \tau)} < 1 ,$$

or

$$c(t, k, \tau) < Rd(t, k, \tau) .$$

Thus we have to show that

$$\begin{aligned}
& \tau - \frac{1 - e^{-k\tau}}{k} e^{-kt} < R\tau - \frac{1 - e^{-k\tau}}{k} e^{-kt} - (R - 1) \frac{1 - e^{-k\tau}}{k} \left[1 + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right] \\
\Leftrightarrow & \tau < R\tau - (R - 1) \frac{1 - e^{-k\tau}}{k} \left[1 + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right] \\
\Leftrightarrow & \tau > \frac{1 - e^{-k\tau}}{k} \left[1 + \frac{1}{2} e^{-2kt} (1 - e^{-k\tau}) \right] \\
\Leftrightarrow & k\tau > (1 - e^{-k\tau}) \left(\frac{3}{2} + \frac{1}{2} e^{-k\tau} \right) \\
\Leftrightarrow & 2k\tau > 3 - 4e^{-k\tau} + e^{-2k\tau} \\
\Leftrightarrow & 2k\tau + 1 > (2 - e^{-k\tau})^2 \\
\Leftrightarrow & \sqrt{2k\tau + 1} > 2 - e^{-k\tau} \text{ for } k\tau > 0.
\end{aligned}$$

We substitute $x = k\tau$ and get

$$\begin{aligned}
\sqrt{2x + 1} & > 2 - e^{-x} \Leftrightarrow \frac{d}{dx} \sqrt{2x + 1} > \frac{d}{dx} [2 - e^{-x}] \\
& \Leftrightarrow \frac{1}{\sqrt{2x + 1}} > e^{-x} \\
& \Leftrightarrow e^x > \sqrt{2x + 1}
\end{aligned}$$

which is true since e^x is convex and $\sqrt{2x + 1}$ is concave. ■

Appendix A.5

Proof of proposition 3.

1) First, we analyze the term $\frac{c(t, k, \tau)}{d(t, k, \tau)}$ for $t = 0$ and $t \rightarrow \infty$.

a) $t \rightarrow \infty$

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{c(t, k, \tau)}{d(t, k, \tau)} &= \frac{\tau}{\tau - \left(1 - \frac{1}{R}\right) \frac{1 - e^{-k\tau}}{k}} \\
&= \frac{k\tau}{k\tau - \left(1 - \frac{1}{R}\right) (1 - e^{-k\tau})} > 1.
\end{aligned}$$

b) $t = 0$

$$\begin{aligned}
\frac{c(0, k, \tau)}{d(0, k, \tau)} &= \frac{\tau - \frac{1 - e^{-k\tau}}{k}}{\tau - \frac{1 - e^{-k\tau}}{k} - \frac{1}{2} \left(1 - \frac{1}{R}\right) \frac{(1 - e^{-k\tau})^2}{k}} \\
&= \frac{k\tau - 1 + e^{-k\tau}}{k\tau - 1 + e^{-k\tau} - \frac{1}{2} \left(1 - \frac{1}{R}\right) (1 - e^{-k\tau})^2} > 1.
\end{aligned}$$

So in order to proof

$$\lim_{t \rightarrow \infty} \frac{c(t, k, \tau)}{d(t, k, \tau)} > \frac{c(0, k, \tau)}{d(0, k, \tau)}$$

we have to show that

$$\frac{k\tau}{k\tau - \left(1 - \frac{1}{R}\right) (1 - e^{-k\tau})} > \frac{k\tau - 1 + e^{-k\tau}}{k\tau - 1 + e^{-k\tau} - \frac{1}{2} \left(1 - \frac{1}{R}\right) (1 - e^{-k\tau})^2}$$

or

$$\begin{aligned}
& \frac{\frac{1}{2} \left(1 - \frac{1}{R}\right) (1 - e^{-k\tau})^2}{k\tau - 1 + e^{-k\tau}} < \frac{\left(1 - \frac{1}{R}\right) (1 - e^{-k\tau})}{k\tau} \\
& \Leftrightarrow \frac{k\tau}{2} (1 - e^{-k\tau}) < k\tau - 1 + e^{-k\tau} \\
& \Leftrightarrow e^{-k\tau} \left(1 + \frac{k\tau}{2}\right) > 1 - \frac{k\tau}{2}.
\end{aligned}$$

This inequality obviously holds for $k\tau \geq 2$. For $k\tau < 2$ we substitute $x = \frac{k\tau}{2}$. Hence we have to show

$$e^{-2x} (1 + x) > 1 - x \Leftrightarrow 1 + x > e^{2x} - xe^{2x}, \text{ for } 0 < x < 1.$$

This follows from Taylor expansion or differentiation.

2) In order to analyze monotonicity and extrema of

$$\theta(t) = \frac{1}{R} \frac{c(t, k, \tau)}{d(t, k, \tau)}$$

we may look at the monotone transformation

$$\begin{aligned}
\eta(t) &= \left(1 - \frac{1}{R}\right)^{-1} \frac{c(t, k, \tau) - d(t, k, \tau)}{d(t, k, \tau)} \\
&= \frac{\frac{1-e^{-k\tau}}{k} [1 - e^{-kt} + \frac{1}{2}e^{-2kt} (1 - e^{-k\tau})]}{\tau - \frac{1-e^{-k\tau}}{k} e^{-kt}} \\
&= \frac{1 - e^{-kt} + \frac{1}{2}e^{-2kt} (1 - e^{-k\tau})}{\frac{k\tau}{1-e^{-k\tau}} - e^{-kt}}.
\end{aligned}$$

Substituting $x = e^{-kt}$ leads to

$$\begin{aligned}
f(x) &= \frac{1 - x + \frac{1}{2} (1 - e^{-k\tau}) x^2}{\frac{k\tau}{1-e^{-k\tau}} - x} \\
&= -\frac{1}{2} (1 - e^{-k\tau}) x + \frac{\left(\frac{k\tau}{2} - 1\right) x + 1}{\frac{k\tau}{1-e^{-k\tau}} - x} \\
&= -\frac{1}{2} (1 - e^{-k\tau}) x + 1 - \frac{k\tau}{2} + \frac{1 - \left(1 - \frac{k\tau}{2}\right) \frac{k\tau}{1-e^{-k\tau}}}{\frac{k\tau}{1-e^{-k\tau}} - x} \\
&= -\frac{1}{2} (1 - e^{-k\tau}) x + 1 - \frac{k\tau}{2} + \frac{(1 - e^{-k\tau}) \left[1 - k\tau + \frac{(k\tau)^2}{2} - e^{-k\tau}\right]}{k\tau - (1 - e^{-k\tau}) x}.
\end{aligned}$$

We need only to show that

a) $f(x)$ is convex for $0 \leq x \leq 1$.

b) $f'(x^*) = 0$ for some $0 < x^* < 1$.

Proof of a):

For $k\tau > 0$ one can easily show by Taylor's theorem that

$$1 - k\tau + \frac{(k\tau)^2}{2} - e^{-k\tau} > 0$$

holds. Now the convexity of $f(x)$ for $0 \leq x \leq 1$ is obvious.

Proof of b):

$$\begin{aligned}
f'(x) &= (1 - e^{-k\tau}) \left[-\frac{1}{2} + \frac{(1 - e^{-k\tau}) \left[1 - k\tau + \frac{(k\tau)^2}{2} - e^{-k\tau} \right]}{(k\tau - (1 - e^{-k\tau})x)^2} \right] \\
f'(0) &= (1 - e^{-k\tau}) \left[-\frac{1}{2} + \underbrace{(1 - e^{-k\tau})}_{<1} \underbrace{\frac{\left[1 - k\tau + \frac{(k\tau)^2}{2} - e^{-k\tau} \right]}{(k\tau)^2}}_{<\frac{1}{2}} \right] < 0 \\
f'(1) &= (1 - e^{-k\tau}) \left[-\frac{1}{2} + \frac{(1 - e^{-k\tau}) \left[1 - k\tau + \frac{(k\tau)^2}{2} - e^{-k\tau} \right]}{(k\tau - (1 - e^{-k\tau}))^2} \right]
\end{aligned}$$

It remains to show that $f'(1) > 0$ or

$$\begin{aligned}
&(k\tau - (1 - e^{-k\tau}))^2 < 2(1 - e^{-k\tau}) \left[1 - k\tau + \frac{(k\tau)^2}{2} - e^{-k\tau} \right] \\
\Leftrightarrow &(k\tau)^2 - 2k\tau(1 - e^{-k\tau}) + (1 - e^{-k\tau})^2 < 2(1 - e^{-k\tau})^2 - 2k\tau(1 - e^{-k\tau}) + (k\tau)^2(1 - e^{-k\tau}) \\
\Leftrightarrow &(k\tau)^2 < (1 - e^{-k\tau})^2 + (k\tau)^2 - (k\tau)^2 e^{-k\tau} \\
\Leftrightarrow &(k\tau)^2 e^{-k\tau} < (1 - e^{-k\tau})^2 \\
\Leftrightarrow &k\tau e^{-0.5k\tau} < 1 - e^{-k\tau}, k\tau > 0 \\
\Leftrightarrow &y e^{-0.5y} < 1 - e^{-y}, y > 0 \\
\Leftrightarrow &y < e^{0.5y} - e^{-0.5y}, y > 0.
\end{aligned}$$

Since this is obviously true, we have shown that $f'(1) > 0$. Due to the continuity of $f'(x)$ there exists an x^* , with $0 < x^* < 1$, such that $f'(x^*) = 0$.

Due to the convexity of $f(x)$ on $[0, 1]$, x^* corresponds to a minimum and we get

$$f'(x) < 0 \text{ for } x < x^*, f'(x) > 0 \text{ for } x > x^*.$$

From $e^{-kT^*} = x^*$ we see that T^* does not depend on R . The monotonicity properties of $\theta(t)$ follow immediately. ■

Appendix A.6

Proof of proposition 4.

1) The formula for the welfare maximizing portfolio $\mathbf{x}$ follows directly from the optimality condition

$$c(\delta, k, \tau) \boldsymbol{\pi} - d(\delta, k, \tau) R V \mathbf{x}^* = \mathbf{0}.$$

2) Next, we show that $d(\delta, k, \tau) > 0$.

We have

$$d(\delta, k, \tau) = \frac{\tau}{\delta} - \left(1 - \frac{1}{R} \right) \frac{1 - e^{-k\tau}}{k} \left[\frac{1}{\delta} + \frac{1}{2} \frac{1}{2k + \delta} (1 - e^{-k\tau}) \right] + \frac{1}{R} \frac{1 - e^{-k\tau}}{k} \frac{1}{k + \delta}.$$

Obviously

$$\frac{\tau}{\delta} > \frac{1 - e^{-k\tau}}{k} \frac{1}{k + \delta}$$

holds.

Since $R > 1$ we have only to show

$$\begin{aligned} \frac{\tau}{\delta} &> \frac{1 - e^{-k\tau}}{k} \left[\frac{1}{\delta} + \frac{1}{2} \frac{1}{2k + \delta} (1 - e^{-k\tau}) \right] \\ \Leftrightarrow k\tau &> (1 - e^{-k\tau}) \left[1 + \frac{1}{2} \frac{\delta}{2k + \delta} (1 - e^{-k\tau}) \right] \\ \Leftrightarrow k\tau &> (1 - e^{-k\tau}) \left[1 + \frac{1}{2} - \frac{1}{2} e^{-k\tau} \right] \\ \Leftrightarrow 2k\tau &> 3 - 4e^{-k\tau} + e^{-2k\tau}. \end{aligned}$$

The last inequality was already proved at the end of appendix A4.

3) Next, we show $c(\delta, k, \tau) > d(\delta, k, \tau)$. One obtains

$$c(\delta, k, \tau) - d(\delta, k, \tau) = \frac{1 - e^{-k\tau}}{k} \left(1 - \frac{1}{R} \right) \left[\frac{1}{\delta} + \frac{1}{2} \frac{1}{2k + \delta} (1 - e^{-k\tau}) - \frac{1}{k + \delta} \right] > 0.$$

4) Finally, it remains to show that

$$c(\delta, k, \tau) < Rd(\delta, k, \tau).$$

One obtains

$$Rd(\delta, k, \tau) - c(\delta, k, \tau) = (R - 1) \left\{ \frac{\tau}{\delta} - \frac{1 - e^{-k\tau}}{k} \left[\frac{1}{\delta} + \frac{1}{2} \frac{1}{2k + \delta} (1 - e^{-k\tau}) \right] \right\} > 0$$

according to part 2 of this appendix. ■

Appendix A.7

Proof of proposition 6.

1) and 2) follow immediately from

$$p(t, \alpha) = \left(\ln \bar{F} + \frac{\mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{k} \right) (1 - e^{-kt}) + z_\alpha \left(\frac{\mathbf{x}^\top V \mathbf{x}}{2k} \right)^{0.5} (1 - e^{-2kt})^{0.5}.$$

3) Since

$$\frac{\partial p(t, \alpha)}{\partial t} = \underbrace{\left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right)}_{>0 \text{ by assumption}} \underbrace{e^{-kt}}_{>0} + z_\alpha \underbrace{\left(\frac{\mathbf{x}^\top V \mathbf{x}}{2k} \right)^{0.5} \frac{ke^{-2kt}}{(1 - e^{-2kt})^{0.5}}}_{>0}$$

we have a sufficient condition for the monotonicity

$$\alpha \geq 0.5 \Leftrightarrow z_\alpha \geq 0 \Rightarrow \frac{\partial p(t, \alpha)}{\partial t} > 0.$$

Since

$$\begin{aligned} \frac{\partial^2 p(t, \alpha)}{\partial t^2} &= -k \underbrace{\left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right)}_{>0 \text{ by assumption}} \underbrace{e^{-kt}}_{>0} \\ &\quad + z_\alpha \underbrace{\left(\frac{\mathbf{x}^\top V \mathbf{x}}{2k} \right)^{0.5}}_{>0} \underbrace{\frac{k^2 e^{-2kt}}{(1 - e^{-2kt})^{1.5}}}_{>0} \underbrace{(e^{-2kt} - 2)}_{<0} \end{aligned}$$

we have a sufficient condition for the concavity

$$\alpha \geq 0.5 \Leftrightarrow z_\alpha \geq 0 \Rightarrow \frac{\partial^2 p(t, \alpha)}{\partial t^2} < 0.$$

4) For the first order condition we have

$$\frac{\partial p(t, \alpha)}{\partial t} = \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) e^{-kt} + z_\alpha \left(\frac{\mathbf{x}^\top V \mathbf{x}}{2k} \right)^{0.5} \frac{ke^{-2kt}}{(1 - e^{-2kt})^{0.5}} = 0$$

or, if $\alpha < 0.5 \Leftrightarrow z_\alpha < 0$,

$$\begin{aligned} & \frac{k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{\left(\frac{\mathbf{x}^\top V \mathbf{x}}{2k} \right)^{0.5}} = \frac{-z_\alpha k e^{-kt^*}}{(1 - e^{-2kt^*})^{0.5}} \\ \Leftrightarrow & \frac{2k \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right)^2}{z_\alpha^2 \mathbf{x}^\top V \mathbf{x}} = \frac{k^2 e^{-2kt^*}}{1 - e^{-2kt^*}} \\ \Leftrightarrow & \frac{2 \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right)^2}{z_\alpha^2 \mathbf{x}^\top V \mathbf{x}} = \frac{k}{e^{2kt^*} - 1} \\ \Leftrightarrow & e^{2kt^*} = \frac{k z_\alpha^2 \mathbf{x}^\top V \mathbf{x}}{2 \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right)^2} + 1 \\ \Leftrightarrow & t^*(\alpha) = \frac{1}{2k} \ln \left[1 + \frac{k z_\alpha^2 \mathbf{x}^\top V \mathbf{x}}{2 \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right)^2} \right] > 0. \end{aligned}$$

For the monotonicity we analyze the inequalities

$$\frac{\partial p(t, \alpha)}{\partial t} = \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right) e^{-kt} + z_\alpha \left(\frac{\mathbf{x}^\top V \mathbf{x}}{2k} \right)^{0.5} \frac{ke^{-2kt}}{(1 - e^{-2kt})^{0.5}} \gtrless 0.$$

Similar to the calculation above, for $\alpha < 0.5 \Leftrightarrow z_\alpha < 0$ we get

$$\begin{aligned} & \frac{k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x}}{\left(\frac{\mathbf{x}^\top V \mathbf{x}}{2k} \right)^{0.5}} \gtrless \frac{-z_\alpha k e^{-kt}}{(1 - e^{-2kt})^{0.5}} \\ \Leftrightarrow & \frac{2k \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right)^2}{z_\alpha^2 \mathbf{x}^\top V \mathbf{x}} \gtrless \frac{k^2 e^{-2kt}}{1 - e^{-2kt}} \\ \Leftrightarrow & \frac{2 \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right)^2}{z_\alpha^2 \mathbf{x}^\top V \mathbf{x}} \gtrless \frac{k}{e^{2kt} - 1} \\ \Leftrightarrow & e^{2kt} \gtrless \frac{k z_\alpha^2 \mathbf{x}^\top V \mathbf{x}}{2 \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right)^2} + 1 \\ \Leftrightarrow & t \gtrless \frac{1}{2k} \ln \left[1 + \frac{k z_\alpha^2 \mathbf{x}^\top V \mathbf{x}}{2 \left(k \ln \bar{F} + \mathbf{x}^\top \boldsymbol{\pi} - \frac{1}{2} \mathbf{x}^\top V \mathbf{x} \right)^2} \right] = t^*(\alpha) \end{aligned}$$

and thus $\frac{\partial p(t, \alpha)}{\partial t} \gtrless 0 \Leftrightarrow t \gtrless t^*(\alpha)$. ■

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