Pension Reform, Retirement and Life-Cycle Unemployment:

Technical Appendix*

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Abstract

This technical note provides a complete model documentation accompanying our paper in Jaag, Keuschnigg and Keuschnigg (2009).

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1 Introduction

We model Austria as a small open economy with an internationally fixed real interest rate. Savings and capital investment result from intertemporal choice with perfect foresight. Households save to ensure smooth consumption in the face of uneven life-cycle income patterns. In particular, they save to top up public pensions and sustain their consumption level during retirement.

The computational model is based on the probabilistic aging approach introduced by Grafenhofer et al. (2006, 2007) which allows for a period length of one year and approximates life-cycle features with a limited number of age states to keep the dimensions of the model low. Our model is a generalization of Gertler (1999) who first introduced a simple life-cycle structure into the basic Blanchard (1995) model. Heijdra, Keuschnigg and Kohler (2004) introduced dynamic search unemployment and derived some analytical results in a Blanchard type model without retirement. Keuschnigg and Keuschnigg (2004) introduced two age groups in this model, workers and retirees, to analyze pension reform. The present model considers eight age groups, allows for mortality in all life-cycle stages, and distinguishes several groups of retirees to take account of the substantial heterogeneity among old and very old generations. The model includes a detailed representation of the pension system in the presence of life-cycle labor supply, retirement and search unemployment.

The birth and mortality patterns mimic the demographic projections in an aging society. The model replicates the continued increase in the retiree worker ratio which is the source of the pension problem and dictates the much discussed changes to the system. Finally, we include in much detail the separate budgets of the public sector and the pension system. The modeling of the pension system includes an individually perceived tax benefit link. We are thus able to calculate the implicit tax component of mandatory contributions and to capture the distortions of the retirement decision as well as labor supply and job search of prime age workers.

2 Overlapping Generations

2.1 Demographics

We define a discrete number of A states of increasing age, and accordingly collect all agents with identical characteristics in the same age group $a \in \{1, \dots, A\}$. People start life in state $a = 1$ with a given set of attributes. The life-cycle characteristics could include a person's earnings potential or her mortality risk but possibly other attributes as well. Aging means that an individual's life-cycle characteristics change when she grows older by switching to state $a + 1$. Although aging shocks arrive stochastically, the average outcome of heterogeneous aging patterns across individuals leads to smooth profiles of expected life-cycle earnings and other characteristics. We measure real time in regular annual periods, with interest, prices and quantities appropriately defined per year. The aging clock runs slower and stochastically.

Households differ not only by their date of birth, but also by their diverse life-cycle histories. An agent's life-cycle history is her biography of aging events that have happened since birth. It is represented by a vector α that records the past dates of aging events. At date t , the set of possible histories of a household that belongs to age group a is

$$\mathcal{N}_t^a \equiv \{(\alpha_1, \dots, \alpha_a) : \alpha_1 < \dots < \alpha_a \leq t\}. \quad (1)$$

A particular life-cycle history is represented by a vector $\alpha \in \mathcal{N}_t^a$. The element α_i , $i \in \{1, \dots, a\}$, denotes the date at which the household who was formerly in age group $i - 1$, became a member of group i . Individual biographies are updated when a person experiences an aging event. Suppose a person is in age group $a - 1$ and is identified by a biography $\alpha = (\alpha_1, \dots, \alpha_{a-1})$. When the next aging shock occurs at the end of period t , she arrives in group a next period. Her biography is appended by the entry $t + 1$ and reads $(\alpha_1, \dots, \alpha_{a-1}, t + 1)$.

To model demographics, we allow for mortality among younger age groups. When an individual with an arbitrarily given life-cycle history plans for next period, she faces the risk of aging and dying. She faces three possible events: (i) she dies with probability $1 - \gamma^a$; (ii) she survives without aging and remains in the same age group with probability $\gamma^a \omega^a$, and (iii) she survives and ages and belongs to age group $a + 1$ next period with probability $\gamma^a (1 - \omega^a)$. With stochastically independent risks, the law of large numbers implies that the individual probabilities for a

certain event correspond to the fraction of people that are subject to this event. To keep track of population heterogeneity, one must carefully identify each agent by her group as well as her aging biography α . The number of agents at date t , in state of life a and with history α is given by $N_{\alpha,t}^a$. Within this group, agents are identical and face the same independent probability of moving to one of the alternative states. The transition process is

$$\begin{aligned} (i) \quad N_{\alpha^\dagger,t+1}^\dagger &= N_{\alpha,t}^a \cdot (1 - \gamma^a), & \text{death,} \\ (ii) \quad N_{\alpha,t+1}^a &= N_{\alpha,t}^a \cdot \gamma^a \omega^a, & \text{no aging,} \\ (iii) \quad N_{\alpha',t+1}^{a+1} &= N_{\alpha,t}^a \cdot \gamma^a (1 - \omega^a), & \text{aging.} \end{aligned} \tag{2}$$

Individuals in the last age group have exhausted the aging process, implying $\omega^A = 1$ as an end condition. They may either survive with probability γ^A within group A or die with probability $1 - \gamma^A$. Observe that only the last age group behaves according to the mortality and demographic assumptions of Blanchard's (1985) perpetual youth model.

2.2 Life-Cycle Optimization

2.2.1 Retirees

Preferences are given by a CES expected utility function, see Farmer (1990) and Weil (1990).

A retired person solves

$$\begin{aligned} V(A_{\alpha,t}^a, P_{\alpha,t}^a) &= \max_{C_{\alpha,t}^a} [(C_{\alpha,t}^a)^\rho + \gamma^a \beta (G \bar{V}_{\alpha,t+1}^a)^\rho]^{1/\rho}, \quad \sigma = \frac{1}{1-\rho}, \\ \bar{V}_{\alpha,t+1}^a &\equiv \omega^a V_{\alpha,t+1}^a + (1 - \omega^a) V_{\alpha',t+1}^{a+1}, \end{aligned} \tag{3}$$

subject to a budget constraint

$$\gamma^a G A_{\alpha,t+1}^a = R_{t+1} (A_{\alpha,t}^a + y_{\alpha,t}^a + z_t^a + i v_t^a - C_{\alpha,t}^a), \quad G P_{\alpha,t+1}^a = G^P P_{\alpha,t}^a, \tag{4}$$

$$y_{\alpha,t}^a = (1 - t^{p,a}) (P_{\alpha,t}^a + p_t^0). \tag{5}$$

A person's expected utility next period, conditional on surviving, is $\bar{V}_{\alpha,t+1}^a$. With probability ω^a , the agent is not aging and expects welfare $V_{\alpha,t+1}^a$ next period. With probability $1 - \omega^a$, she ages and expects welfare $V_{\alpha',t+1}^{a+1}$. In this event, the agent's biography must be updated from α to α' . Retiree groups $a < A$ are subject to aging risk while the last group A is not ($\omega^A = 1$), leading to $\bar{V}_{\alpha,t+1}^A \equiv V_{\alpha,t+1}^A$. The subjective discount factor is β , the survival probability is

γ^a and the constant intertemporal elasticity of substitution is σ . Net of tax pension income per capita y_α^a consists of an earnings related part P_α^a and a flat, lump-sum part p^0 . Retirees spend on current consumption C_α^a which is partly financed from previously accumulated financial assets A_α^a . Income transfers are z^a , and iv^a are intergenerational transfers (outflow of bequests if negative, and received inheritances if positive). We assume the existence of reverse life-insurance contracts as in the OLG literature based on Blanchard (1985).

Pension Indexation: The growth of pension entitlements includes a policy choice with respect to benefit indexation (factor $G^P = 1 + g^P$) which might be anywhere between wage and price indexation. Full wage indexation ($G^P = G$) means that pensions after retirement grow in line with general wage growth. Price indexation ($G^P = 1$ and $g^P = 0$) means that benefits remain constant in real terms so that pension benefits grow slower than wage earnings during the retirement period and living standards of pensioners relative to prime age workers is eroded.¹

To solve for the optimal consumption policy, it is useful to define shadow prices

$$\eta_{\alpha,t}^a \equiv \frac{dV_{\alpha,t}^a}{dA_{\alpha,t}^a}, \quad \bar{\eta}_{\alpha,t}^a \equiv \omega^a \eta_{\alpha,t}^a + (1 - \omega^a) \eta_{\alpha',t}^{a+1}, \quad (6)$$

$$\lambda_{\alpha,t}^a \equiv \frac{dV_{\alpha,t}^a}{dP_{\alpha,t}^a}, \quad \bar{\lambda}_{\alpha,t}^a = \omega^a \lambda_{\alpha,t}^a + (1 - \omega^a) \lambda_{\alpha',t}^{a+1}. \quad (7)$$

Optimality and envelope conditions are

$$(C_{\alpha,t}^a) : (C_{\alpha,t}^a)^{\rho-1} = \beta R_{t+1} (G \bar{V}_{\alpha,t+1}^a)^{\rho-1} \cdot \bar{\eta}_{t+1}^a, \quad (8)$$

$$(A_{\alpha,t}^a) : \eta_t^a = \beta R_{t+1} (G \bar{V}_{\alpha,t+1}^a / V_{\alpha,t}^a)^{\rho-1} \cdot \bar{\eta}_{t+1}^a, \quad (9)$$

$$(P_{\alpha,t}^a) : \lambda_t^a = \beta (G \bar{V}_{\alpha,t+1}^a / V_{\alpha,t}^a)^{\rho-1} \cdot [(1 - t^{p,a}) R_{t+1} \bar{\eta}_{t+1}^a + \gamma^a G^P \bar{\lambda}_{t+1}^a]. \quad (10)$$

Define the term Ω^a which can be related to the MRS across age groups,

$$\Omega^a \equiv \omega^a + (1 - \omega^a) (\Lambda^a)^{1-\rho}, \quad \Lambda^a \equiv \frac{V^{a+1}/C^{a+1}}{V^a/C^a}. \quad (11)$$

To get the modified Euler equation, use $dV^a/dA^a = \eta^a = (V^a/C^a)^{1-\rho}$ from (8) and (9). Take out η^a from (6) and use the last result to get $\bar{\eta}^a = \Omega^a (V^a/C^a)^{1-\rho}$ with Ω^a given in (11). Writing

¹When nominal pensions $\tilde{P}_t$ grow annually with the factor G^P , we have $\tilde{P}_{t+1} = G^P \tilde{P}_t$. Dividing by efficiency units which grow by $X_{t+1} = GX_t$, and noting $\tilde{P}_t = X_t P_t$, we get $G P_{t+1} = G^P P_t$ as in (4), where P is pension per efficiency unit.

expected utility in (3) as $\bar{V}^a = [\omega^a C^a + (1 - \omega^a) \Lambda^a C^{a+1}] (V^a / C^a)$ results, upon substitution, in $\bar{\eta}^a = [\omega^a C^a + (1 - \omega^a) \Lambda^a C^{a+1}]^{\rho-1} (\bar{V}^a)^{1-\rho} \Omega^a$. Using this in (8) yields the modified Euler equation where $\sigma = 1 / (1 - \rho)$,

$$G \left[\omega^a C_{\alpha,t+1}^a + (1 - \omega^a) \Lambda_{t+1}^a C_{\alpha',t+1}^{a+1} \right] = (\beta R_{t+1} \Omega_{t+1}^a)^\sigma \cdot C_{\alpha,t}^a. \quad (12)$$

Proposition 1 (Retiree Policy) *Consumption $C_{\alpha,t}^a$ and indirect utility $V_{\alpha,t}^a$ are*

$$\begin{aligned} (i) \quad C_{\alpha,t}^a &= (1/\Delta_t^a) \cdot (A_{\alpha,t}^a + S_{\alpha,t}^a + T_{\alpha,t}^a), \quad \sigma = 1/(1 - \rho), \\ (ii) \quad V_{\alpha,t}^a &= (\Delta_t^a)^{1/\rho} C_{\alpha,t}^a, \\ (iii) \quad \Delta_t^a &= 1 + \gamma^a \beta^\sigma (R_{t+1} \Omega_{t+1}^a)^{\sigma-1} \Delta_{t+1}^a, \\ (iv) \quad \Omega_{t+1}^a &= \omega^a + (1 - \omega^a) (\Lambda_{t+1}^a)^{1-\rho}, \\ &\quad \Lambda_{t+1}^a = (\Delta_{t+1}^{a+1} / \Delta_{t+1}^a)^{1/\rho}, \\ (v) \quad S_{\alpha,t}^a &= (1 - t^{p,a}) P_{\alpha,t}^a + \gamma^a G \bar{S}_{\alpha,t+1}^a / (R_{t+1} \Omega_{t+1}^a), \\ (vi) \quad \bar{S}_{\alpha,t+1}^a &= \omega^a S_{\alpha,t+1}^a + (1 - \omega^a) (\Lambda_{t+1}^a)^{1-\rho} S_{\alpha',t+1}^{a+1}, \\ (vii) \quad T_{\alpha,t}^a &= z_t^a + i v_t^a + (1 - t^{p,a}) p_t^0 + \gamma^a G \bar{T}_{\alpha,t+1}^a / (R_{t+1} \Omega_{t+1}^a), \\ (viii) \quad \bar{T}_{\alpha,t+1}^a &= \omega^a T_{\alpha,t+1}^a + (1 - \omega^a) (\Lambda_{t+1}^a)^{1-\rho} T_{\alpha',t+1}^{a+1}. \end{aligned} \quad (13)$$

The marginal propensity to consume is $1/\Delta_t^a$ and S and T are pension and transfer wealth. End conditions for the last age group are $\omega^A = \Omega_{t+1}^A = 1$, $\bar{S}_{\alpha,t+1}^A = S_{\alpha,t+1}^A$ and $\bar{T}_{\alpha,t+1}^A = T_{\alpha,t+1}^A$.

Proof. We first show that the consumption function indeed fulfills the Euler equation and is thus the optimal policy. Note that (13.ii) together with (11) implies Λ^a as in (13.iv). Since (13.iii-iv) depend only on the interest rate and other factors independent of α , the terms Δ_t^a and by implication Λ_t^a and Ω_t^a are all independent of α . Now insert (13.i) into the l.h.s. of (12), use $A_{\alpha,t+1}^a = A_{\alpha',t+1}^{a+1}$, collect terms, note the definitions of Ω_{t+1}^a and $\bar{S}_{\alpha,t+1}^a$, and get $[\Omega_{t+1}^a A_{\alpha,t+1}^a + \bar{S}_{\alpha,t+1}^a + \bar{T}_{\alpha,t+1}^a] G / \Delta_{t+1}^a = (\beta R_{t+1} \Omega_{t+1}^a)^\sigma C_{\alpha,t}^a$. Multiply by $\Delta_{t+1}^a / (R_{t+1} \Omega_{t+1}^a)$ and use (13.v, vii) and (4) on the l.h.s.: $A_{\alpha,t}^a + S_{\alpha,t}^a + T_{\alpha,t}^a - C_{\alpha,t}^a = \gamma^a \beta^\sigma (R_{t+1} \Omega_{t+1}^a)^{\sigma-1} \Delta_{t+1}^a C_{\alpha,t}^a$. Substitute again (13.i) on the l.h.s. and cancel $C_{\alpha,t}^a$. The result corresponds exactly to (13.iii). Hence, the stated policy is optimal since it satisfies (12) which is merely a reformulation of the necessary conditions in (8-10).

Second, show that (13.ii) is indirect utility as it identically fulfills the Bellmann equation. Replace consumption in the Euler equation by (13.ii). Multiply by $(\Delta_{t+1}^a)^{1/\rho}$, use definitions of

Λ_{t+1}^a and $\bar{V}_{\alpha,t+1}^a$ and get $G\bar{V}_{\alpha,t+1}^a = (\beta R_{t+1} \Omega_{t+1}^a)^\sigma V_{\alpha,t}^a (\Delta_{t+1}^a / \Delta_t^a)^{1/\rho}$. Take the power of ρ , write $\rho\sigma = \sigma - 1$, multiply by $\gamma^a \beta$, and use (13.iii) to substitute $\gamma^a \beta^\sigma (R_{t+1} \Omega_{t+1}^a)^{\sigma-1} \Delta_{t+1}^a = \Delta_t^a - 1$. The result is $\gamma^a \beta (G\bar{V}_{\alpha,t+1}^a)^\rho = (\Delta_t^a - 1) \cdot (V_{\alpha,t}^a)^\rho / \Delta_t^a$. By (13.ii), $(V_{\alpha,t}^a)^\rho / \Delta_t^a = (C_{\alpha,t}^a)^\rho$, which is used on the r.h.s. Rearrange to see that the Bellmann equation (3) is identically fulfilled. ■

According to proposition 1, a retiree's optimal policy is to spend at each date a fraction of life-time resources on current consumption C . Wealth consists of her previously accumulated financial assets A plus the present value of future pension entitlements S plus transfer wealth T . Reflecting mortality risk, older people have a higher marginal propensity to consume out of life-time wealth than younger ones. If all had the same survival probability $\gamma^a = \gamma$, the marginal propensity to consume would be the same for all age groups.

Proposition 1 also contains the case of the last age group. When in the last group, agents cannot age any more, implying $\omega^A = 1$. As a consequence, $\Omega^A = 1$, $\bar{S}_\alpha^A = S_\alpha^A$ and $\bar{T}_\alpha^A = T_\alpha^A$. Next we show that pension wealth, consisting of the present value of earnings related pensions, is the product of the stock of pension entitlements times its shadow price. Note, however, that earnings related pension wealth $S_{\alpha,t}^a$ is only part of total pension wealth which also includes the present value of lump-sum pensions p_t^0 per capita as part of $T_{\alpha,t}^a$.

Proposition 2 (*Pension Wealth*) *Earnings related pension wealth is $S_{\alpha,t}^a = \tilde{\lambda}_t^a P_{\alpha,t}^a$, where $\tilde{\lambda}_t^a \equiv \lambda_t^a / \eta_t^a$. Lump-sum pension wealth, together with other variables independent of history, is part of transfer wealth $T_{\alpha,t}^a$.*

Proof. As a first step, we show

$$\frac{\bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a} = \frac{\tilde{\lambda}_{t+1}^a}{\Omega_{t+1}^a}, \quad \bar{\lambda}_{t+1}^a \equiv \omega^a \tilde{\lambda}_{t+1}^a + (1 - \omega^a) (\Lambda_{t+1}^a)^{1-\rho} \tilde{\lambda}_{t+1}^{a+1}. \quad (i)$$

Writing out $\bar{\lambda}_{t+1}^a$ and noting $\bar{\eta}^a = \eta^a \Omega^a$ by the results following (11) yields, upon suppressing some indices,

$$\frac{\bar{\lambda}^a}{\bar{\eta}^a} \Omega^a = \omega^a \frac{\lambda^a}{\eta^a} + (1 - \omega^a) \frac{\lambda^{a+1}}{\eta^{a+1}} \cdot \frac{\eta^{a+1}}{\eta^a}. \quad (ii)$$

By the results following (11) together with (13.ii), we have $\eta^a = (V^a / C^a)^{1-\rho} = (\Delta^a)^{(1-\rho)/\rho}$. Using Λ^a as in (13.iv) yields $\eta^{a+1} / \eta^a = (\Lambda^a)^{1-\rho}$. Substituting this into the last term in (ii) and using $\lambda^a / \eta^a = \tilde{\lambda}^a$ proves (i).

To prove the proposition, multiply the last envelope condition by $1/\eta_t^a$ and use the first envelope condition (9) together with the results in (i) yields

$$\tilde{\lambda}_t^a = 1 - t^{p,a} + \frac{\gamma^a G^P \bar{\lambda}_{t+1}^a}{R_{t+1} \Omega_{t+1}^a}. \quad (\text{iii})$$

From this equation, it is evident that $\tilde{\lambda}^a$ is independent of history α . Multiply by $P_{\alpha,t}^a$, note $GP_{\alpha,t+1}^a = G^P P_{\alpha,t}^a$ by (5) and use $P_{\alpha,t+1}^a = P_{\alpha',t+1}^{a+1}$ to get

$$\tilde{\lambda}_t^a P_{\alpha,t}^a = (1 - t^{p,a}) P_{\alpha,t}^a + \frac{\gamma^a G \left[\omega^a \tilde{\lambda}_{t+1}^a P_{\alpha,t+1}^a + (1 - \omega^a) (\Lambda_{t+1}^a)^{1-\rho} \tilde{\lambda}_{t+1}^{a+1} P_{\alpha',t+1}^{a+1} \right]}{\Omega_{t+1}^a R_{t+1}}. \quad (\text{iv})$$

This equation yields exactly the same solution for $\tilde{\lambda}_t^a P_{\alpha,t}^a$ as the system (13.v-vi) yields for $S_{\alpha,t}^a$. The two solutions being equal establishes the result. ■

2.2.2 Workers

A strength of the model is that it is able to capture life-cycle profiles of earnings and unemployment risks. We assume that agents are endowed with a labor productivity (efficiency units) θ^a that changes across age groups but is constant within each group and is time invariant. This parameter reflects the life-cycle profile of wage earnings. With respect to unemployment risk, an individual of age group a is assumed to have, at the beginning of each period, a job without searching with an exogenous probability $1 - \varepsilon^a$. The law of large numbers implies that a mass $(1 - \varepsilon^a) N^a$ of the labor force instantaneously gets a job. The other $\varepsilon^a N^a$ have to search and are matched with probability f per unit of search effort ζ^a . As a result, a share u^a of age group a is unemployed and a share $1 - u^a$ gets a job,

$$u^a = (1 - \zeta^a f) \varepsilon^a, \quad 1 - u^a = (1 - \varepsilon^a) + \varepsilon^a \zeta^a f, \quad (14)$$

where f is individually taken as given from the labor market and is the same for all groups.

In any given period, workers pursue a sequence of activities: (i) At the beginning of period, workers must search with probability ε^a , and are immediately allocated a job with probability $1 - \varepsilon^a$. In the first event, they supply a search effort ζ^a with effort cost $\varphi_S(\zeta) \theta^a$. Job seekers are matched with an open vacancy in case of successful job search, leading to a probability of

employment $1 - u^a$ and of unemployment u^a as above. (ii) If unemployed, agents stay idle. If they have a job, they choose total hours worked l^a , incurring an effort cost $\varphi(l^a)\theta^a$.²

In choosing consumption, savings and work related activities today, agents anticipate how this affects their welfare during later life-cycle stages. Expected utility of a prime age worker is

$$\begin{aligned} V(A_{\alpha,t}^a, P_{\alpha,t}^a) &= \max_{Q, \zeta, l} [(Q_{\alpha,t}^a)^\rho + \gamma^a \beta (G\bar{V}_{\alpha,t+1}^a)^\rho]^{1/\rho}, \\ \bar{V}_{\alpha,t+1}^a &\equiv \omega^a V_{\alpha,t+1}^a + (1 - \omega^a) V_{\alpha',t+1}^{a+1}, \\ Q_{\alpha,t}^a &= C_{\alpha,t}^a - \bar{\varphi}_t^a, \\ \bar{\varphi}_t^a &= [\varepsilon^a \varphi_S(\zeta_t^a) + (1 - u_t^a) \varphi_L(l_t^a)] \theta^a. \end{aligned} \tag{15}$$

The problem is stated per capita. In assuming preferences that are additively separable in consumption C and job related efforts, we eliminate income effects on labor supply. It will be verified later that all efforts differ by age state but are independent of aging history such that $l_{\alpha,t}^a = l_t^a$. This symmetry is essential for aggregation.

Workers accumulate assets according to $\gamma^a G A_{\alpha,t+1}^a = R_{t+1} (A_{\alpha,t}^a + y_t^a + z_t^a + iv_t^a - C_{\alpha,t}^a)$. Wage related income consists of net of tax salaries and unemployment benefits,

$$\begin{aligned} y_t^a &\equiv [(1 - u_t^a) l_t^a w_t^{n,a} + u_t^a b_t^a] \theta^a, \quad u_t^a \equiv (1 - \zeta_t^a f_t) \cdot \varepsilon^a, \\ b_t^a &= b^U \cdot l_t^a w_t^{n,a} + b_0^a, \quad w_t^{n,a} = (1 - t^{w,a} - t^{s,a}) \cdot w_t^a, \end{aligned} \tag{16}$$

where $w_t^{n,a}$ is the net hourly wage per efficiency/productivity unit, and θ^a is the agent's productivity (skill). The observed wage of an agent with skill θ^a is $\theta^a w^a$. Agents receive an unemployment benefit b_t^a (per efficiency unit) which may be partly indexed to net wage income and partly be constant (such as with social assistance to the long-term unemployed). Full indexation would be $b_t^a = b^U \cdot l_t^a w_t^{n,a}$ with $b_0^a = 0$, no indexation is $b^U = 0$ and $b_t^a = b_0^a$. Of course, parameters must be such that the total benefit b_t^a reflects an empirically realistic replacement rate. The indexation rule for setting unemployment benefits is very important in determining the unemployment response to tax changes.

²Since life-cycle productivity θ^a is exogenous, the multiplication of effort costs is only a scaling that becomes useful when allowing for training and skill formation to endogenize wage profiles.

It will be useful to slightly rewrite wage related income

$$\begin{aligned}
y_t^a &= (1 - u_t^a + b^U u_t^a) l_t^a w_t^{n,a} \theta^a + u_t^a \cdot b_0^a \theta^a, \quad du_t^a / d\zeta_t^a = -\varepsilon^a f_t, \\
\gamma^a G A_{\alpha,t+1}^a &= R_{t+1} [A_{\alpha,t}^a + y_t^a - \bar{\varphi}_t^a + z_t^a + i v_t^a - Q_{\alpha,t}^a], \\
G P_{\alpha,t+1}^a &= R_{t+1}^P [m^a \cdot (1 - u_t^a + b^1 u_t^a) l_t^a w_t^a \theta^a + P_{\alpha,t}^a].
\end{aligned} \tag{17}$$

When switching to the next group, stocks are $A_{\alpha',t+1}^{a+1} = A_{\alpha,t+1}^a$ and $P_{\alpha',t+1}^{a+1} = P_{\alpha,t+1}^a$. The PAYG system could pay a notional interest R_{t+1}^P on past entitlements. The factor m^a gives the yearly increase of the determination base for pensions. Wage income also raises pension entitlements during a share b^1 of unemployment periods (approximately 70% in Austria). It is not the unemployment benefit that raises the pension assessment base, but rather the last gross earnings $lw\theta$ prior to unemployment (assessment base for unemployment benefits).

Shadow prices η_t^a and λ_t^a are forward looking, i.e. independent of history,

$$\begin{aligned}
\eta_t^a &\equiv \frac{\partial V_{\alpha,t}^a}{\partial A_{\alpha,t}^a}, \quad \bar{\eta}_{t+1}^a \equiv \omega^a \eta_{\alpha,t+1}^a + (1 - \omega^a) \eta_{\alpha,t+1}^{a+1}, \\
\lambda_t^a &\equiv \frac{\partial V_{\alpha,t}^a}{\partial P_{\alpha,t}^a}, \quad \bar{\lambda}_{t+1}^a \equiv \omega^a \lambda_{\alpha,t+1}^a + (1 - \omega^a) \lambda_{\alpha',t+1}^{a+1}.
\end{aligned} \tag{18}$$

The optimality conditions for (15) subject to (17) are (see below for some explanations)

$$\begin{aligned}
Q_{\alpha,t}^a : \quad (Q_{\alpha,t}^a)^{\rho-1} &= \beta R_{t+1} (G \bar{V}_{\alpha,t+1}^a)^{\rho-1} \cdot \bar{\eta}_{t+1}^a, \\
\zeta_t^a : \quad \varphi'_S(\zeta_t^a) &= f_t \cdot [\Gamma_t^a w_t^a l_t^a - \varphi_L(l_t^a) - b_0^a], \\
\Gamma_t^a &\equiv (1 - b^U) (1 - t^{w,a} - t^{s,a}) + (1 - b^1) m^a \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a}, \\
l_t^a : \quad \varphi'_L(l_t^a) &= (1 - \hat{t}^{w,a} - \hat{t}^{s,a}) w_t^a, \\
\hat{t}^{w,a} &\equiv t^{w,a} - b^U \cdot \frac{u^a}{1-u^a} (1 - t^{w,a} - t^{s,a}), \\
\hat{t}^{s,a} &\equiv t^{s,a} - m^a \cdot \left(1 + b^1 \frac{u^a}{1-u^a}\right) \cdot \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a}.
\end{aligned} \tag{19}$$

Note how labor supply and search effort are strengthened by the tax benefit link m^a . By way of contrast, b^1 creates an additional benefit of remaining unemployed and weakens search effort. The labor supply condition is obtained from rearranging

$$(1 - u^a) \varphi'_L = (1 - u^a + b^U u^a) (1 - t^{w,a} - t^{s,a}) w^a + m^a (1 - u^a + b^1 u^a) w^a \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a}.$$

Divide by $1 - u^a$, rearrange and group together to get the condition above. Note that $\frac{u^a}{1-u^a}$ converts unemployment benefits per unit of working time. The effective tax rate is reduced since, due to partial indexation, part of increased work effort also raises unemployment compensation.

This extra benefit is subtracted from the statutory tax to give an effective rate. An increase in wage income also raises future pensions which reduces the effective contribution tax.

The condition for search effort collects in Γ^a all terms that are proportional to gross wage income $w^a l^a$. The term Γ^a captures gains and losses when an individual moves from unemployment into employment by accepting a job that pays a gross wage $w^a l^a$. The first term shows the increase in net income after taxes when accepting a job offer but this gain is reduced by the factor $1 - b^U$ due to the foregone unemployment benefit (the part that is proportional to wage income). The second term in Γ^a shows the gain in terms of higher pension rights resulting from the job salary. Again, this gain is reduced by a factor $1 - b^1$ due to the foregone pension rights that stem from unemployment periods also counting towards pensions.

The envelope theorem yields:

$$\begin{aligned} A_{\alpha,t}^a : \quad \eta_t^a &= (G\bar{V}_{\alpha,t+1}^a/V_{\alpha,t}^a)^{\rho-1} \beta R_{t+1} \cdot \bar{\eta}_{t+1}^a, \\ P_{\alpha,t}^a : \quad \lambda_t^a/\eta_t^a &= R_{t+1}^P \gamma^a \bar{\lambda}_{t+1}^a / (R_{t+1} \bar{\eta}_{t+1}^a). \end{aligned} \quad (20)$$

There are no current dividends since pension payments start only upon retirement.

The next step is to get the modified Euler equation. Define

$$\Omega_{\alpha,t}^a \equiv \omega^a + (1 - \omega^a) (\Lambda_{\alpha,t}^a)^{1-\rho}, \quad \Lambda_{\alpha,t}^a \equiv \frac{V_{\alpha,t}^{a+1}/Q_{\alpha,t}^{a+1}}{V_{\alpha,t}^a/Q_{\alpha,t}^a}. \quad (21)$$

Get $dV_{\alpha,t}^a/dA_{\alpha,t}^a = \eta_{\alpha,t}^a = (V_{\alpha,t}^a/Q_{\alpha,t}^a)^{1-\rho}$ by combining conditions for Q and A . Take out η^a from $\bar{\eta}_{\alpha,t+1}^a$ and use the last result to get $\bar{\eta}_{\alpha,t+1}^a = \Omega_{\alpha,t+1}^a (V_{\alpha,t+1}^a/Q_{\alpha,t+1}^a)^{1-\rho}$ with Ω^a given in (21). Writing expected utility in (15) as $\bar{V}_{\alpha,t+1}^a = \left[\omega^a Q_{\alpha,t+1}^a + (1 - \omega^a) \Lambda_{\alpha,t+1}^a Q_{\alpha,t+1}^{a+1} \right] (V_{\alpha,t+1}^a/Q_{\alpha,t+1}^a)$ results, upon substitution, in $\bar{\eta}_{\alpha,t+1}^a = \left[\omega^a Q_{\alpha,t+1}^a + (1 - \omega^a) \Lambda_{\alpha,t+1}^a Q_{\alpha,t+1}^{a+1} \right]^{\rho-1} (\bar{V}_{\alpha,t+1}^a)^{1-\rho} \Omega_{\alpha,t+1}^a$. Using this in the first envelope condition (20) yields the Euler equation with $\sigma = 1/(1 - \rho)$,

$$G \left[\omega^a Q_{\alpha,t+1}^a + (1 - \omega^a) \Lambda_{\alpha,t+1}^a Q_{\alpha,t+1}^{a+1} \right] = (\beta R_{t+1} \Omega_{\alpha,t+1}^a)^\sigma \cdot Q_{\alpha,t}^a. \quad (22)$$

The following closed form solution for consumption and welfare is obtained.

Proposition 3 (Younger Workers) Consumption $Q_{\alpha,t}^a$ and indirect utility $V_{\alpha,t}^a$ are

$$\begin{aligned}
(a) \quad Q_{\alpha,t}^a &= (1/\Delta_t^a) (A_{\alpha,t}^a + H_{\alpha,t}^a + S_{\alpha,t}^a + T_{\alpha,t}^a), \\
(b) \quad V_{\alpha,t}^a &= (\Delta_t^a)^{1/\rho} Q_{\alpha,t}^a, \quad \sigma = 1/(1-\rho), \\
(c) \quad \Delta_t^a &= 1 + \gamma^a \beta^\sigma (R_{t+1} \Omega_{t+1}^a)^{\sigma-1} \Delta_{t+1}^a, \\
(d) \quad \Omega_{t+1}^a &= \omega^a + (1-\omega^a) (\Lambda_{t+1}^a)^{1-\rho}, \quad \Lambda_t^a = (\Delta_t^{a+1}/\Delta_t^a)^{1/\rho}, \\
(e) \quad H_{\alpha,t}^a &= y_t^a - \bar{\varphi}_t^a + \bar{s}_t^a + G \bar{H}_{\alpha,t+1}^a / (R_{t+1} \Omega_{t+1}^a), \\
(f) \quad \bar{H}_{\alpha,t+1}^a &= \omega^a H_{\alpha,t+1}^a + (1-\omega^a) (\Lambda_{t+1}^a)^{1-\rho} H_{\alpha',t+1}^{a+1}, \\
(g) \quad S_{\alpha,t}^a &= -\bar{s}_t^a + \gamma^a G \bar{S}_{\alpha,t+1}^a / (R_{t+1} \Omega_{t+1}^a), \\
&\quad \bar{s}_t^a \equiv m^a \cdot (1 - u_t^a + b^1 u_t^a) w_t^a l_t^a \theta^a \cdot R_{t+1}^P \gamma^a \bar{\lambda}_{t+1}^a / (R_{t+1} \bar{\eta}_{t+1}^a), \\
(h) \quad \bar{S}_{\alpha,t+1}^a &= \omega^a S_{\alpha,t+1}^a + (1-\omega^a) (\Lambda_{t+1}^a)^{1-\rho} S_{\alpha',t+1}^{a+1}, \\
(i) \quad T_{\alpha,t}^a &= z_t^a + i v_t^a + \gamma^a G \bar{T}_{\alpha,t+1}^a / (R_{t+1} \Omega_{t+1}^a), \\
(j) \quad \bar{T}_{\alpha,t+1}^a &= \omega^a T_{\alpha,t+1}^a + (1-\omega^a) (\Lambda_{t+1}^a)^{1-\rho} T_{\alpha',t+1}^{a+1}.
\end{aligned} \tag{23}$$

Proof. (i) We show that the consumption function indeed fulfills the Euler equation and is thus the optimal policy. Note that (23.b) and (21) imply Λ^a as in (23.d). Since (23.c-d) depend only on interest and other factors independent of α , the terms Δ_t^a and in turn Λ_t^a and Ω_t^a are independent of α . Now insert (23.a) into the l.h.s. of (22), use $A_{\alpha,t+1}^a = A_{\alpha',t+1}^{a+1}$, collect terms, note the definitions of Ω_{t+1}^a and $\bar{H}_{\alpha,t+1}^a$, and get $[\Omega_{t+1}^a A_{\alpha,t+1}^a + \bar{H}_{\alpha,t+1}^a + \bar{S}_{\alpha,t+1}^a + \bar{T}_{\alpha,t+1}^a] G / \Delta_{t+1}^a = (\beta R_{t+1} \Omega_{t+1}^a)^\sigma Q_{\alpha,t}^a$. Multiply by $\Delta_{t+1}^a / (R_{t+1} \Omega_{t+1}^a)$ and use (17) and (23.e-j) on the left side, $A_{\alpha,t}^a + H_{\alpha,t}^a + S_{\alpha,t}^a + T_{\alpha,t}^a - Q_{\alpha,t}^a = \gamma^a \beta^\sigma (R_{t+1} \Omega_{t+1}^a)^{\sigma-1} \Delta_{t+1}^a Q_{\alpha,t}^a$. Substitute again (23.a) on the l.h.s. and cancel $Q_{\alpha,t}^a$. The result corresponds to (23.c). Hence, the stated policy is optimal since it satisfies (22) which is merely a reformulation of the necessary conditions in (19-20).

(ii) We show that (23.b) gives indirect utility as it identically fulfills the Bellmann equation. Again, start with the Euler equation and replace consumption terms by (23.b). Multiply the result by $(\Delta_{t+1}^a)^{1/\rho}$, use the definitions of Λ_{t+1}^a and $\bar{V}_{\alpha,t+1}^a$ and get $G \bar{V}_{\alpha,t+1}^a = (\beta R_{t+1} \Omega_{t+1}^a)^\sigma \cdot V_{\alpha,t}^a (\Delta_{t+1}^a / \Delta_t^a)^{1/\rho}$. Next, take the power of ρ , write $\rho\sigma = \sigma - 1$, multiply by $\gamma^a \beta$, and use (23.c) to substitute $\gamma^a \beta^\sigma (R_{t+1} \Omega_{t+1}^a)^{\sigma-1} \Delta_{t+1}^a = \Delta_t^a - 1$. The result is $\gamma^a \beta (G \bar{V}_{\alpha,t+1}^a)^\rho = (\Delta_t^a - 1) \cdot (V_{\alpha,t}^a)^\rho / \Delta_t^a$. By (23.b), $(V_{\alpha,t}^a)^\rho / \Delta_t^a = (Q_{\alpha,t}^a)^\rho$, which is used on the r.h.s. A minor rearrangement shows that the Bellmann equation (15) is identically fulfilled. ■

The factor $\Omega^a > 1$ reflects the individual valuation of the aging risk and leads to an increased discount rate of young workers. The implicit, effective contribution rate is $\hat{t}^s$ while t^s is the

statutory rate. The presence of a tax benefit link implies $\hat{t}^s < t^s$ in (19). Intuitively, an agent anticipates that earning higher wage income today adds to her pension claims and raises the pension in retirement. This extra benefit corresponds to the part $t^s - \hat{t}^s$ of her contribution payment while only the part $\hat{t}^s$ is considered a tax without any corresponding benefit. This implicit tax reflects the fact that contributions are forced retirement savings that earn a lower rate of return than savings invested at the market rate of interest.

Proposition 4 (*Pension Wealth*) *Earnings related pension wealth is $S_{\alpha,t}^a \equiv \tilde{\lambda}_t^a P_{\alpha,t}^a$, where $\tilde{\lambda}_t^a \equiv \lambda_t^a / \eta_t^a$. The value of future lump-sum pensions is reflected in $T_{\alpha,t}^a$.*

Proof. By the same steps as in Prop. 2, using $\bar{\eta}^a = \eta^a \Omega^a$, we can write

$$\frac{\bar{\lambda}_{\alpha,t+1}^a}{\bar{\eta}_{\alpha,t+1}^a} = \frac{\bar{\lambda}_{t+1}^a}{\Omega_{t+1}^a}, \quad \bar{\lambda}_{t+1}^a \equiv \omega^a \tilde{\lambda}_{t+1}^a + (1 - \omega^a) (\Lambda^a)^{1-\rho} \tilde{\lambda}_{t+1}^{a+1}. \quad (\text{i})$$

Use (i) and write $\bar{s}_t^a$ in (23.g) as

$$\bar{s}_t^a = m^a \cdot (1 - u_t^a + b^1 u_t^a) l_t^a w_t^a \theta^a \cdot \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\Omega_{t+1}^a}. \quad (\text{ii})$$

Use $\tilde{\lambda}_t^a \equiv \lambda_t^a / \eta_t^a$ in the envelope condition for P in (20), and the result in (i),

$$\tilde{\lambda}_t^a = R_{t+1}^P \gamma^a \bar{\lambda}_{t+1}^a / (\Omega_{t+1}^a R_{t+1}). \quad (\text{iii})$$

The shadow price $\tilde{\lambda}$ is forward looking and thus independent of history. Multiply (iii) by $P_{\alpha,t}^a$, use (17.c) to replace $R_{t+1}^P P_{\alpha,t}^a$, use (ii) and write out $\bar{\lambda}_{t+1}^a$ in the remaining term. Noting $P_{\alpha,t+1}^a = P_{\alpha',t+1}^{a+1}$ in this last term yields

$$\tilde{\lambda}_t^a P_{\alpha,t}^a = -\bar{s}_t^a + \gamma^a G \frac{\omega^a \cdot \tilde{\lambda}_{t+1}^a P_{\alpha,t+1}^a + (1 - \omega^a) (\Lambda^a)^{1-\rho} \cdot \tilde{\lambda}_{t+1}^{a+1} P_{\alpha',t+1}^{a+1}}{\Omega_{t+1}^a R_{t+1}}. \quad (\text{iv})$$

The result follows upon noting that the solution of $S_{\alpha,t}^a$ in (24.g-h) is identical to the solution of $\tilde{\lambda}_t^a P_{\alpha,t}^a$ in this equation. ■

Pension entitlements are augmented by today's wage income on account of the tax benefit link, giving rise to the earnings related part $S_{\alpha,t}^a$ of pension wealth. During old age, workers also receive lump-sum pensions p_t^0 per capita that are unrelated to today's earnings. The value of this flat pension is reflected in transfer wealth $T_{\alpha,t}^a$ in (23.i) which is, by expected wealth $\bar{T}_{\alpha,t+1}^a$, ultimately connected to the value of $T_{\alpha,t}^a$ in retirement as stated in (13.vii).

Definition 1 (Effective Tax Rates) The effective tax rates on intensive labor supply and job search are defined in (19) by $\varphi'_L(l_t^a) = (1 - \tau_{L,t}^a) w_t^a$ and $\varphi'_S(\zeta_t^a) = f_t \cdot \left[(1 - \tau_{S,t}^a) w_t^a l_t^a - \varphi_{L,t}^a \right]$, giving

$$\tau_{L,t}^a \equiv \hat{t}^{w,a} + \hat{t}^{s,a}, \quad \tau_{S,t}^a = 1 - \Gamma_t^a + b_0^a / w_t^a l_t^a.$$

In the absence of government and social security, $\Gamma^a = 1$ and $\tau_S^a = \tau_L^a = 0$, leaving the laissez-faire conditions $\varphi'_L(l^a) = w^a$ and $\varphi'_S(\zeta^a) = f \cdot [w^a l^a - \varphi_L^a]$.

2.2.3 Mixed Age Group

The mixed partially retired age group is special since it consists of workers and retirees. The composition depends on the endogenous retirement decision. Productivity θ drops to zero upon retirement. Active labor income is replaced with pensions. We approximate endogenous retirement by assuming that agents in the mixed group work a fraction x_t of their time endowment and spend a fraction $1 - x_t$ in retirement, giving average income $\bar{y}_{\alpha,t}^a = x_t y_t^a + (1 - x_t) (1 - t^{p,a}) (P_{\alpha,t}^a + p_t^0)$. When people retire later, i.e. continue to work longer, they obtain a larger share of work income but also incur a progressively increasing disutility $\varphi_R(x_t)$, $\varphi_R', \varphi_R'' > 0$.

The large heterogeneity in previously accumulated pension entitlements, $P_{\alpha,t}^a$, creates heterogeneity in the retirement decision x . This heterogeneity must be avoided to allow for analytical aggregation. We thus assume a two stage decision structure. In a first stage, people collectively decide on a uniform retirement date x . This is, of course, an approximation which successfully captures retirement incentives and yet retains symmetry. In stage two, people decide on job search and work during their active part of the mixed phase, and remain idle when unemployed. We solve backwards and first derive work related choices.

Taking the retirement date as given, agents maximize expected life-time utility

$$V(A_{\alpha,t}^a, P_{\alpha,t}^a) = \max_{Q_{\alpha,t}^a, \zeta_t^a, l_t^a} \left[(Q_{\alpha,t}^a)^\rho + \gamma^a \beta (G \bar{V}_{\alpha,t+1}^a)^\rho \right]^{1/\rho}, \quad (24)$$

where

$$\begin{aligned} \bar{V}_{\alpha,t+1}^a &\equiv \omega^a V_{\alpha,t+1}^a + (1 - \omega^a) V_{\alpha',t+1}^{a+1}, \\ Q_{\alpha,t}^a &= C_{\alpha,t}^a - x_t \cdot \bar{\varphi}_t^a - \varphi_R(x_t), \\ \bar{\varphi}_t^a &= [\varepsilon^a \varphi_S(\zeta_t^a) + (1 - u_t^a) \varphi_L(l_t^a)] \theta^a, \\ u_t^a &= \varepsilon^a (1 - \zeta_t^a) f_t. \end{aligned} \quad (25)$$

It is now extremely important to correctly state wage related income consisting of an average of work and retirement income. We define employed income exactly the same way as in (16-17). To retain symmetry in x_t , we need to have identical transfers z_t^a within each group. Writing asset accumulation as $\gamma^a G A_{\alpha,t+1}^a = R_{t+1} (A_{\alpha,t}^a + \bar{y}_{\alpha,t}^a + z_t^a + i v_t^a - C_{\alpha,t}^a)$ and referring to (17), it is useful to define average income per efficiency unit as

$$\begin{aligned}\bar{y}_{\alpha,t}^a &= x_t \cdot y_t^a + (1 - x_t) \cdot (1 - t^{p,a}) (P_{\alpha,t}^a + p_t^0), \\ y_t^a &= [(1 - u_t^a + b^U u_t^a) l_t^a w_t^{n,a} + u_t^a b_0^a] \theta^a, \\ w_t^{n,a} &= (1 - t^{w,a} - t^{s,a}) w_t^a, \\ u_t^a &= (1 - \zeta_t^a f_t) \varepsilon^a, \quad du_t^a / d\zeta_t^a = -\varepsilon^a f_t.\end{aligned}\tag{26}$$

Expand the savings equation by effort costs. The state variables thus evolve as

$$\begin{aligned}\gamma^a G A_{\alpha,t+1}^a &= R_{t+1} [A_{\alpha,t}^a + \bar{y}_{\alpha,t}^a - x_t \bar{\varphi}_t^a - \varphi_R(x_t) + z_t^a + i v_t^a - Q_{\alpha,t}^a], \\ G P_{\alpha,t+1}^a &= R_{t+1}^P [\mu_w(x_t) m^a (1 - u_t^a + b^1 u_t^a) w_t^a l_t^a \theta^a + \mu_p(x_t) P_{\alpha,t}^a], \\ \mu_w(x_t) &= (x_t - x^R) \cdot \mu^1 + \mu_w^2, \quad \mu_p(x_t) = (x_t - x^R) \cdot \mu^1 + \mu_p^2.\end{aligned}\tag{27}$$

Wage related income $\bar{y}_{\alpha}^a$ is an average of work and retirement income. The weights x_t and $1 - x_t$ correspond to the fraction of time spent working and in retirement. Pension entitlements $P_{\alpha,t+1}^a$ next period are equal to today's stock plus some additions. They grow with new entitlements in the amount of $\mu_w m^a$ times current wage income. The terms $\mu_p(x)$ and $\mu_p(x)$ capture institutional retirement incentives which compensate for continued contributions and foregone pensions when retirement is postponed. If pension rights are insensitive to a change in retirement age, $\mu^1 = 0$, implying constant $\mu_w = \mu_w^2$ and $\mu_p = \mu_p^2$. For example $\mu_w^2 = x^0$ may be equal to the initial participation rate to reflect that agents work only part of the time during the mixed age state and therefore accumulate new entitlements only during this active time span, while $\mu_p^2 = 1$ assures accumulation without any pension supplements or discounts. If $\mu^1 > 0$, postponed retirement creates additional pension supplements a la Gruber and Wise while early retirement before the statutory retirement age x^R results in pension discounts. This formulation also allows to analyze the impact of raising statutory retirement age x^R . An increase in statutory retirement age means that the full regular pension is attained only when retiring at a later date. If people continue to retire at the same date, this means a reduction in pension income as μ_w and μ_p would become smaller.

Shadow prices are defined as in (18). The optimality conditions are

$$\begin{aligned}
(Q_{\alpha,t}^a)^{\rho-1} &= \beta R_{t+1} (G\bar{V}_{\alpha,t+1}^a)^{\rho-1} \cdot \bar{\eta}_{t+1}^a, \\
\varphi'_L(l_t^a) &= (1 - \hat{t}^{w,a} - \hat{t}^{s,a}) w_t^a, \\
\hat{t}^{w,a} &\equiv t^{w,a} - b^U \frac{u^a}{1 - u^a} (1 - t^{w,a} - t^{s,a}), \\
\hat{t}^{s,a} &\equiv t^{s,a} - m^a \cdot \left(1 + b^1 \frac{u^a}{1 - u^a}\right) \frac{\mu_w(x)}{x} \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a}, \\
\varphi'_S(\zeta_t^a) &= f_t \cdot [\Gamma_t^a w_t^a l_t^a - \varphi_L(l_t^a) - b_0^a], \\
\Gamma_t^a &\equiv (1 - b^U) (1 - t^{w,a} - t^{s,a}) + m^a (1 - b^1) \frac{\mu_w(x)}{x} \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a},
\end{aligned} \tag{28}$$

and the envelope conditions are

$$\begin{aligned}
\eta_t^a &= \beta R_{t+1} \bar{\eta}_{t+1}^a \cdot (G\bar{V}_{\alpha,t+1}^a / V_{\alpha,t}^a)^{\rho-1}, \\
\frac{\lambda_t^a}{\eta_t^a} &= (1 - x_t) (1 - t^{p,a}) + \mu_p(x_t) \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a}.
\end{aligned} \tag{29}$$

Except for pension benefits during part $1 - x$ of the mixed age state and the participation incentives μ , the problem of the mixed group yields exactly the same solution as for younger workers. Since the necessary conditions for Q and A are the same, the Euler equation remains as in (22). To prove Proposition 3 for the mixed group, we restate the laws of motion in (23):

Proposition 5 (*Consumption Mixed Group*) *Consumption is given in (23) with*

$$\begin{aligned}
H_{\alpha,t}^a &= x_t \cdot (y_t^a - \bar{\varphi}_t^a) + \mu_w(x_t) \bar{s}_t^a + \frac{G\gamma^a \bar{H}_{\alpha,t+1}^a}{R_{t+1} \bar{\Omega}_{t+1}^a}, \\
\bar{H}_{\alpha,t+1}^a &= \omega^a H_{\alpha,t+1}^a, \quad H_{\alpha',t+1}^{a+1} = 0, \\
S_{\alpha,t}^a &= (1 - x_t) (1 - t^{p,a}) P_{\alpha,t}^a - \mu_w(x_t) \bar{s}_t^a + \frac{G\gamma^a \bar{S}_{\alpha,t+1}^a}{R_{t+1} \bar{\Omega}_{t+1}^a}, \\
T_t^a &= z_t^a + i v_t^a + (1 - x_t) (1 - t^{p,a}) p_t^0 - \varphi_R^a(x_t) + \frac{G\gamma^a \bar{T}_{t+1}^a}{R_{t+1} \bar{\Omega}_{t+1}^a}.
\end{aligned} \tag{30}$$

where $\bar{s}_t^a$, $\bar{S}_{\alpha,t+1}^a$ and $\bar{T}_{t+1}^a$ as in (23).

Proof. Completely parallel to Proposition 3 (Younger Workers), using (30) instead. When an agent switches to full retirement in age group $a + 1$, human wealth vanishes completely while the lump-sum pension is then included in T_{t+1}^{a+1} as defined in (13.vii) which adds to the expected value of $\bar{T}_{t+1}^a$ in (30). ■

Proposition 6 (Pension Wealth) Earnings related pension wealth is $S_{\alpha,t}^a \equiv \tilde{\lambda}_t^a P_{\alpha,t}^a$, where $\tilde{\lambda}_t^a \equiv \lambda_t^a / \eta_t^a$. The current value of lump-sum pensions consumed by the mixed group is included in T_t^a while those consumed in states of full retirement are capitalized in $\bar{T}_{t+1}^a$.

Proof. By the same steps as before, we can write

$$\frac{\bar{\lambda}_{\alpha,t+1}^a}{\bar{\eta}_{\alpha,t+1}^a} = \frac{\bar{\lambda}_{t+1}^a}{\Omega_{t+1}^a}, \quad \bar{\lambda}_{t+1}^a \equiv \omega^a \tilde{\lambda}_{t+1}^a + (1 - \omega^a) (\Lambda^a)^{1-\rho} \tilde{\lambda}_{t+1}^{a+1}. \quad (i)$$

Now use $\tilde{\lambda}_t^a \equiv \lambda_t^a / \eta_t^a$ to rewrite the last envelope condition and use (i),

$$\tilde{\lambda}_t^a = (1 - x_t) (1 - t^{p,a}) + \mu_p(x_t) \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\Omega_{t+1}^a}. \quad (ii)$$

Multiply by $P_{\alpha,t}^a$, replace $R_{t+1}^P \mu_{p,t} P_{\alpha,t}^a$ by (27) and use the definition of $\bar{s}_t^a$ in (23.h) together with (i). In the resulting equation, write out $\bar{\lambda}_{t+1}^a$ and note $P_{\alpha,t+1}^a = P_{\alpha',t+1}^{a+1}$ for the newly aging,

$$\begin{aligned} \tilde{\lambda}_t^a P_{\alpha,t}^a &= (1 - x_t) (1 - t^{p,a}) P_{\alpha,t}^a - \mu_w(x_t) \cdot \bar{s}_t^a \\ &: + \frac{\gamma^a G \left[\omega^a \cdot \tilde{\lambda}_{t+1}^a P_{\alpha,t+1}^a + (1 - \omega^a) (\Lambda_{t+1}^a)^{1-\rho} \cdot \tilde{\lambda}_{t+1}^{a+1} P_{\alpha',t+1}^{a+1} \right]}{R_{t+1} \Omega_{t+1}^a}. \end{aligned} \quad (iii)$$

The result follows upon noting that the solution of $S_{\alpha,t}^a$ in (30) is identical to the solution of $\tilde{\lambda}_t^a P_{\alpha,t}^a$ in this equation. ■

2.2.4 Endogenous Retirement

Consider first *individual* retirement incentives. Maximizing (24) yields a f.o.c.

$$\begin{aligned} \frac{dV_{\alpha,t}^a}{dx_t} &= \beta \gamma^a R_{t+1} \bar{\eta}_{t+1}^a (G \bar{V}_{\alpha,t+1}^a / V_{\alpha,t}^a)^{\rho-1} \cdot [Z_{\alpha,t}^a - \varphi'_R(x_t)] \geq 0, \\ Z_{\alpha,t}^a &\equiv y_t^a - \bar{\varphi}_t^a - (1 - t^{p,a}) (P_{\alpha,t}^a + p_t^0) \\ &: + [\mu'_w(x) m^a (1 - u_t^a + b^1 u_t^a) w_t^a l_t^a \theta^a + \mu'_p(x) P_{\alpha,t}^a] \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a}. \end{aligned} \quad (31)$$

Interpretation is straightforward. The variable $Z_{\alpha,t}^a$ identifies four consequences of retiring one instant later. (i) earn $y_t^a - \bar{\varphi}_t^a$ for an additional period, (ii) give up pension $(1 - t^{p,a}) (P_{\alpha,t}^a + p_t^0)$ for that period, (iii) pay contributions for another period and get higher pensions in all future periods on account of the tax benefit link (μ'_w times first term in square bracket), and (iv) get an increase in future pensions to compensate foregone current pension (term $\mu'_p P_{\alpha,t}^a$). Of course,

the last two terms are present only if the system gives positive participation incentives $\mu'(x) > 0$ in the sense of Gruber and Wise.

Unfortunately, the optimality condition $Z_{\alpha,t}^a = \varphi'_R(x_t)$ cannot be implemented since the heterogeneity in pension entitlements $P_{\alpha,t}^a$ prevents a symmetric retirement date. Symmetric retirement is necessary for aggregation. It is therefore assumed that retirement is chosen on average (coordinated in the semi-retired group). Instead of maximizing (24), the problem becomes

$$\max_{x_t} \sum_{\alpha} [(Q_{\alpha,t}^a)^{\rho} + \gamma^a \beta (G\bar{V}_{\alpha,t+1}^a)^{\rho}]^{1/\rho} \cdot N_{\alpha,t}^a. \quad (32)$$

Obviously, we get a f.o.c. as in (31) for each individual type α . Using the first condition in (28) together with (23.b), we can write it as $dV_{\alpha,t}^a/dx_t = (\Delta_t^a)^{1/\rho-1} \cdot [Z_{\alpha,t}^a - \varphi'_R(x_t)]$, where the multiplicative term is independent of history. To maximize the joint objective, multiply by $N_{\alpha,t}^a$ on both sides and take the sum $\sum_{\alpha}$, resulting in $\sum_{\alpha} [N_{\alpha,t}^a Z_{\alpha,t}^a - \varphi'_R(x_t) N_{\alpha,t}^a] = 0$. Noting the aggregation procedure below, we have $\sum_{\alpha} N_{\alpha}^a = N^a$ and $\bar{P}^a = P^a/N^a$. Thus, the jointly optimal retirement date is

$$\begin{aligned} \varphi'_R(x_t) &= y_t^a - \bar{\varphi}_t^a - (1 - t^{p,a}) (\bar{P}_t^a + p_t^0) \\ &: + [\mu'_w \cdot m^a (1 - u_t^a + b^1 u_t^a) w_t^a l_t^a \theta^a + \mu'_p \cdot \bar{P}_t^a] \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a}. \end{aligned} \quad (33)$$

The interpretation is straightforward and corresponds exactly to equation (31) above, except that it applies on average.

To define the implicit tax rate t^R on continued work (retirement tax), we first take the simplest case of fixed labor supply l^a without job search, implying zero effort costs $\bar{\varphi}^a$ and eliminating any unemployment rate. Suppose further that all dynamic linkages due to the accumulation of earnings related pension points are cut out, implying $P_{\alpha}^a = 0$ and leaving only a flat pension p^0 per capita. Let all taxes be zero except for a pension contribution rate t^s . One is left with $y^a = (1 - t^s) w^a l^a$ and $\bar{y}^a = x y^a + (1 - x) p^0$. In this simplest case, we can write the retirement decision as $\varphi'_R(x) = (1 - t^s) w^a l^a \theta^a - p^0 = (1 - t^R) w^a l^a \theta^a$ where $t^R \equiv t^s + p^0/(w^a l^a \theta^a)$ is the implicit retirement tax.³ It is the sum of contribution rate t^s and pension replacement rate $p^0/(w^a l^a \theta^a)$ and can, thus, become very high. This retirement tax corresponds to a labor market participation tax in the sense of Immervoll et al. (2007).

³In this simple case, there is no heterogeneity in individual pension entitlements, so that “coordination” is not needed. The retirement decision follows from individual optimization.

In our very detailed model, the implicit retirement tax is more involved. To build intuition, we first consider retirement without fiscal policy when $t^s, t^w, \bar{P}, p^0, b^1, b^U, \mu, \mu'$ are all zero,

$$\varphi'_R(x_t) = (1 - u_t^a) l_t^a w_t^a \theta^a - \bar{\varphi}_t^a.$$

We now define an implicit retirement tax, relative to $(1 - u^a) l^a w^a \theta^a$, such that when it is zero, this laissez faire condition is reproduced. Substituting work related income y^a , the definition of $\bar{s}^a$ and unemployment benefits b^a , and rewriting (33) yields

$$\begin{aligned} \varphi'_R(x_t) &= (1 - u_t^a) l_t^a w_t^a \theta^a - \bar{\varphi}_t^a \\ &: - [(t^{w,a} + t^{s,a}) l_t^a w_t^a \cdot (1 - u_t^a) - b_t^a \cdot u_t^a] \theta^a \\ &: - (1 - t^{p,a}) (\bar{P}_t^a + p_t^0) + \mu'_{w,t} \cdot \bar{s}_t^a + \mu'_{p,t} \cdot \bar{P}_t^a \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a}. \end{aligned}$$

The interpretation is clear. The first line lists the net gains of working longer in the absence of government. The second line reflects the extra net taxes paid when retiring later, consisting of wage tax and contribution rate minus foregone unemployment benefits. The third line lists foregone pensions in the first term while the last two terms reflect retirement incentives consisting of pension supplements when retirement is postponed (Gruber Wise incentives μ'). Expressing all terms relative to expected labor income $(1 - u_t^a) l_t^a w_t^a \theta^a$ yields the implicit retirement tax,

$$\begin{aligned} \varphi'_R(x_t) &= (1 - \tau_{R,t}^a) (1 - u_t^a) l_t^a w_t^a \theta^a - \bar{\varphi}_t^a \\ \tau_{R,t}^a &= t^{w,a} + t^{s,a} - \frac{u_t^a b_t^a \theta^a}{(1 - u_t^a) l_t^a w_t^a \theta^a} \\ &: + \frac{(1 - t^{p,a}) (\bar{P}_t^a + p_t^0) - \mu'_{w,t} \cdot \bar{s}_t^a - \mu'_{p,t} \cdot \bar{P}_t^a \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a}}{(1 - u_t^a) l_t^a w_t^a \theta^a}, \\ \bar{s}_t^a &= m^a \cdot (1 - u_t^a + b^1 u_t^a) w_t^a l_t^a \theta^a \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a}. \end{aligned} \tag{34}$$

The implicit tax rate t^R on continued work consists of the sum of statutory tax rates $t^w + t^s$ minus unemployment benefits consumed, plus the net loss in retirement income. This net loss consists of the instantaneous foregone net of tax pensions. However, the last two terms indicate that this loss is reduced by the increase in future pension benefits on account of the Gruber/Wise mechanism for increased actuarial fairness. The present value of these future gains are captured by the shadow price $\bar{\lambda}/\bar{\eta}$. This implicit retirement tax summarizes all the disincentives for

postponing retirement that are built into the pension system. In the absence of any government activity, the implicit tax is zero, $\tau_{R,t}^a = 0$. When there is no unemployment ($u = 0$), no earnings related pension (i.e. no tax-benefit link, $\hat{t}^s = t^s$ and $\bar{P}^a = 0$), and no other taxes ($t^p = t^w = 0$), and if there is no pension adjustment relating to early or late retirement ($\mu = \mu' = 0$), then $\tau_{R,t}^a = t^{s,a} + p^0 / (w^a l^a \theta^a)$ as before.

Definition 2 (Effective Tax Rates) *The effective tax rates on intensive labor supply and job search are defined by $\varphi'_L(l_t^a) = (1 - \tau_{L,t}^a) w_t^a$, $\varphi'_S(\zeta_t^a) = f_t \cdot \left[(1 - \tau_{S,t}^a) w_t^a l_t^a - \varphi_{L,t}^a \right]$, and $\varphi'_R(x) = (1 - \tau_R^a) (1 - u^a) l^a w^a \theta^a - \bar{\varphi}^a$, giving*

$$\tau_{L,t}^a \equiv \hat{t}^{w,a} + \hat{t}^{s,a}, \quad \tau_{S,t}^a = 1 - \Gamma_t^a + b_0^a / w_t^a l_t^a,$$

and $\tau_{R,t}^a$ in (34). In the absence of government and social security, $\Gamma = 1$ and $\tau_S = \tau_L = 0$, leaving the laissez-faire conditions for hours worked, $\varphi'_L(l^a) = w^a$, job search, $\varphi'_S(\zeta^a) = f \cdot [w^a l^a - \varphi_L^a]$, and retirement $\varphi'_R(x) = (1 - u^a) l^a w^a \theta^a - \bar{\varphi}^a$.

2.3 Household Sector Aggregation

2.3.1 Demographic Structure

At date t , age group a includes a number $N_{\alpha,t}^a$ of people with the same biography α . This is the smallest unit of identical agents. The total number of people in age group a is obtained by adding up over all possible histories α ending up in this group a ,

$$N_t^a \equiv \sum_{\alpha \in \mathcal{N}_t^a} N_{\alpha,t}^a, \quad N_t \equiv \sum_{a \leq A} N_t^a. \quad (35)$$

The aggregation formula takes the sum over all possible biographies with varying dates of birth that could conceivably lead to age group a in period t . Given identical death and aging probabilities within a given group, one can now use the law of large numbers for analytical aggregation. The aggregate population groups evolve deterministically over time where $N_{t+1,t+1}^1$ refers to the mass of newborns who arrive at the beginning of period $t+1$ in the first age group.

Proposition 7 (Demographic Structure) *The demographic laws of motion are*

$$\begin{aligned}
(i) \quad & N_{t+1}^a = \gamma^a \omega^a \cdot N_t^a + \gamma^{a-1} (1 - \omega^{a-1}) \cdot N_t^{a-1}, \\
(ii) \quad & N_{t+1}^1 = \gamma^1 \omega^1 \cdot N_t^1 + N_{t+1,t+1}^1, \\
(iii) \quad & N_{t+1} = N_t + N_{t+1,t+1}^1 - \sum_{a \leq A} (1 - \gamma^a) N_t^a,
\end{aligned} \tag{36}$$

where the last group satisfies $\omega^A = 1$.

Proof. The smallest homogeneous population unit is $N_{\alpha,t}^a$. It is assumed large enough so that the law of large numbers applies. It decomposes into three groups. In any given period, members of group a can either stay in the same group, move to group $a + 1$ next period, or die. Conversely, agents in group a next period come from stayers $\alpha \in \mathcal{N}_t^a$ and movers from the younger group $\mathcal{N}_t^{a-1}$. Obviously, a stayer still belongs to group a next period, $\alpha \in \mathcal{N}_{t+1}^a$, while a mover belongs to $\mathcal{N}_t^{a-1}$ this period and $\alpha' \in \mathcal{N}_{t+1}^a$ next period. A mover gets another entry $t + 1$ in her biography, $\alpha' = (\alpha_1, \dots, \alpha_{a-1}, t + 1)$. By definition, group a next period can be decomposed into incumbents and new arrivers. Using (2),

$$\begin{aligned}
\sum_{\alpha \in \mathcal{N}_{t+1}^a} N_{\alpha,t+1}^a &= \sum_{\alpha \in \mathcal{N}_t^a} \gamma^a \omega^a N_{\alpha,t}^a + \sum_{\alpha \in \mathcal{N}_t^{a-1}} \gamma^{a-1} (1 - \omega^{a-1}) N_{\alpha,t}^a, \\
N_{t+1}^a &= \gamma^a \omega^a N_t^a + \gamma^{a-1} (1 - \omega^{a-1}) N_t^{a-1}.
\end{aligned} \tag{a}$$

Transition probabilities are identical for all members of the same group and independent of past history. They can thus be moved in front of the sum operators. Since the last group cannot age further, the event in (2.ii) is precluded, implying the restriction $\omega^A = 1$. In the first group, the inflow is the number of newborns n_t who start life with a biography $\alpha' = (t + 1)$. Adding up (36.i-ii) and noting the restriction $\omega^A = 1$ yields (36.iii). ■

The key demographic parameters are the birth rate, the transition rates to successive age groups, and the mortality rates. Since all are exogenous, the demographic subsystem evolves autonomously as in (36.a-c). The demographic steady state results from the requirement that inflows and outflows of any age group must balance to yield constant group size. Using (36.a-b),

$$N^1 = \frac{N_{t,t}^1}{1 - \gamma^1 \omega^1}, \quad N^a = \frac{\gamma^{a-1} (1 - \omega^{a-1})}{1 - \gamma^a \omega^a} \cdot N^{a-1}. \tag{37}$$

The stationary size of group 1 is determined by the exogenous inflow of newborns. The long-run magnitude of other groups and of the total population results upon recursively applying

the second equation. For any demographic transition, the exogenous driving force is the inflow $N_{t,t}^1$ of newborns. With this flow exogenously specified and constant, the system arrives at a stationary population N .⁴

Workers and retirees are entirely different and must be distinguished very carefully. In the mixed group, we have introduced the share of time x allocated to work and the share $1 - x$ to retirement. We have associated this with a part x of that group being active workers and the remaining part being retirees. This is extended by introducing the vector x^a which takes values $x^a = 1$ for fully active groups, $0 < x^a < 1$ for the mixed group, and $x^a = 0$ for a fully retired group. With this notation, we have

$$N^{w,a} = x^a N^a, \quad N^{r,a} = (1 - x^a) N^a, \quad N^a = N^{w,a} + N^{r,a}. \quad (38)$$

Obviously then, $N^{w,a} = 0$ for a fully retired group with $a > a^m$ (a^m being the index for the partially retired, mixed group), and $N^{r,a} = 0$ for $a < a^m$. Only in the mixed group, both workers and retirees coexist. Taking account of zeros and ones in the vector x , we get the aggregate number of workers and retirees as $N^W = x'N = \sum_a x^a N^a$ and $N^R = (1 - x)'N = \sum_a (1 - x^a) N^a$, respectively. It is obvious that $N^W + N^R = N$.

2.3.2 Accumulation of Pension Claims

Pension claims are a stock variable with a law of motion which is repeated, for convenience, from (17), (27) and (4),

$$\begin{aligned} a < a^m : GP_{\alpha,t+1}^a &= R_{t+1}^P [m^a (1 - u_t^a + b^1 u_t^a) l_t^a w_t^a \theta^a + P_{\alpha,t}^a], \\ a = a^m : GP_{\alpha,t+1}^a &= R_{t+1}^P [\mu_{w,t} m^a (1 - u_t^a + b^1 u_t^a) w_t^a l_t^a \theta^a + \mu_{p,t} P_{\alpha,t}^a], \\ a > a^m : GP_{\alpha,t+1}^a &= G^P P_{\alpha,t}^a. \end{aligned} \quad (39)$$

We start with $P_{t+1}^a = \sum_{\alpha \in \mathcal{N}_t^a} P_{\alpha,t+1}^a N_{\alpha,t+1}^a + \sum_{\alpha' \in \mathcal{N}_t^{a-1}} P_{\alpha',t+1}^a N_{\alpha',t+1}^a$ for fully active workers where again $\alpha' = (\alpha, t + 1)$. Multiply by G , use $N_{\alpha,t+1}^a = \gamma^a \omega^a N_{\alpha,t}^a$ and (39) in the first term and apply the aggregator $P_t^a = \sum_{\alpha} P_{\alpha,t}^a N_{\alpha,t}^a$. Taking out terms that do not depend on history α yields the first term below. In the second term, the mass of movers with biographies $\alpha' = (\alpha, t + 1)$ is

⁴For details on the empirical implementation on real population data see Grafenhofer et al. (2006).

a fraction $\gamma^{a-1} (1 - \omega^{a-1})$ of all $\alpha \in \mathcal{N}_t^{a-1}$. We thus substitute $N_{\alpha',t+1}^a = \gamma^{a-1} (1 - \omega^{a-1}) N_{\alpha,t}^{a-1}$. Using $P_{\alpha',t+1}^a = P_{\alpha,t+1}^{a-1}$ and again plugging in $GP_{\alpha,t+1}^{a-1}$ from (39) yields

$$\begin{aligned} GP_{t+1}^a &= \gamma^a \omega^a [B_t^a + \delta_t^a P_t^a] + \gamma^{a-1} (1 - \omega^{a-1}) [B_t^{a-1} + \delta_t^{a-1} P_t^{a-1}], \\ B_t^a &\equiv R_{t+1}^P \mu_{w,t}^a m^a \cdot (1 - u_t^a + b^1 u_t^a) l_t^a w_t^a \theta^a \cdot N_t^a, \\ \delta_t^a &\in \{R_{t+1}^P, R_{t+1}^P \mu_p(x_t), G^P\}, \quad \mu_{w,t}^a \in \{1, \mu_w(x_t), 0\}. \end{aligned} \quad (40)$$

Newly acquired entitlements are given by B_t^a which is obviously zero for fully retired agents with $\mu_w^a = 0$, $a > a^m$. The factor δ^a determines growth of pension entitlements where $\delta_t^a = R_{t+1}^P$ for $a < a^m$ (fully active groups). The mixed group ($a = a^m$) has $\delta_t^a = R_{t+1}^P \mu_p(x_t)$ because of pension discounts and supplements on account of early and late retirement. Fully retired groups ($a > a^m$) have $\delta^a = G^P$ since pension grow by the rate of g^P after retirement to allow pensioners to (partially) participate in general wage growth. Since there are no initial pension claims for new labor market entrants, entitlements of the first group are not increased by an inflow of a predecessor, giving $GP_{t+1}^1 = \gamma^1 \omega^1 [B_t^1 + \delta_t^1 P_t^1]$.

2.3.3 Accumulation of Assets

Denote aggregate values of transfers, bequests and pensions received by

$$Z_t^a = z_t^a N_t^a, \quad IV_t^a = iv_t^a N_t^a, \quad P_t^a \equiv \sum_{\alpha} P_{\alpha,t}^a N_{\alpha,t}^a = \bar{P}_t^a N_t^a. \quad (41)$$

Total pensions are $P^a + p^0 N^a$ and also include a flat pension independent of any tax benefit link, giving a per capita pension $\bar{P}^a + p^0$, $\bar{P}^a = P^a / N^a$, see also section 2.2.4.

Earlier analysis has shown that the terms $\bar{\varphi}_t^a$, y_t^a and $\bar{s}_t^a$ are all independent of individual histories α . Aggregate wage related income $\bar{y}_{\alpha,t}^a$ in (26) is

$$Y_t^a = \sum_{\alpha} \bar{y}_{\alpha,t}^a N_{\alpha,t}^a = x_t y_t^a N_t^a + (1 - x_t) (1 - t^{p,a}) (P_t^a + p_t^0 N_t^a) = \bar{y}_t^a N_t^a. \quad (42)$$

It will be convenient to define income per capita as $\bar{y}_t^a$,

$$\begin{aligned} \bar{y}_t^a &\equiv Y_t^a / N_t^a = x_t \cdot y_t^a + (1 - x_t) \cdot (1 - t^{p,a}) (\bar{P}_t^a + p_t^0), \\ y_t^a &= [(1 - u_t^a + b^U u_t^a) w_t^{n,a} l_t^a + u_t^a b_0^a] \theta^a, \quad u_t^a = (1 - \zeta_t^a f_t) \varepsilon^a. \end{aligned} \quad (43)$$

Aggregate income $Y_t^a = \bar{y}_t^a N_t^a$ is therefore $Y_t^a = y_t^a N_t^a$ for fully active agents with $x = 1$ and $Y_t^a = (1 - t^{p,a}) (P_t^a + p_t^0 N_t^a)$ for fully retired persons. People in the mixed group get $Y_t^a = \bar{y}_t^a N_t^a$ which depends on the retirement date x .

For convenience, we repeat individual savings,

$$\gamma^a G A_{\alpha,t+1}^a = R_{t+1} S_{\alpha,t}^a, \quad S_{\alpha,t}^a \equiv A_{\alpha,t}^a + \bar{y}_{\alpha,t}^a + z_t^a + iv_t^a - C_{\alpha,t}^a. \quad (44)$$

Aggregating and using (41-43) yields [note $Y_t^a + Z_t^a + IV_t^a = (\bar{y}_t^a + z_t^a + iv_t^a) N_t^a$]

$$A_t^a \equiv \sum_{\alpha \in \mathcal{N}_t^a} A_{\alpha,t}^a N_{\alpha,t}^a, \quad S_t^a = A_t^a + Y_t^a + Z_t^a + IV_t^a - C_t^a. \quad (45)$$

As before, the aggregate value of next period's assets of group a consists of two parts,

$$A_{t+1}^a = \sum_{\alpha \in \mathcal{N}_t^a} A_{\alpha,t+1}^a N_{\alpha,t+1}^a + \sum_{\alpha \in \mathcal{N}_t^{a-1}} A_{\alpha',t+1}^a N_{\alpha',t+1}^a, \quad \alpha' = (\alpha, t+1).$$

There are $N_{\alpha,t+1}^a = \gamma^a \omega^a N_{\alpha,t}^a$ stayers and $N_{\alpha',t+1}^a = \gamma^{a-1} (1 - \omega^{a-1}) N_{\alpha,t}^{a-1}$ movers. Multiply by G and substitute these flows together with (44),

$$G A_{t+1}^a = \omega^a R_{t+1} \sum_{\alpha \in \mathcal{N}_t^a} S_{\alpha,t}^a N_{\alpha,t}^a + (1 - \omega^{a-1}) R_{t+1} \sum_{\alpha \in \mathcal{N}_t^{a-1}} S_{\alpha,t}^{a-1} N_{\alpha,t}^{a-1}.$$

In the second term we have also used $A_{\alpha',t+1}^a = A_{\alpha,t+1}^{a-1}$. Applying the aggregator (45) yields⁵

$$\begin{aligned} G A_{t+1}^a &= R_{t+1} [\omega^a S_t^a + (1 - \omega^{a-1}) S_t^{a-1}], \\ G A_{t+1}^1 &= R_{t+1} \omega^1 S_t^1, \quad S_t^a = A_t^a + Y_t^a + Z_t^a + IV_t^a - C_t^a. \end{aligned} \quad (46)$$

The inflow into the first group is from the newborn who start life without assets. Summing over all age groups and using $\omega^A = 1$ gives

$$G A_{t+1} = R_{t+1} \sum_a S_t^a. \quad (47)$$

In the aggregate, all given bequests must be equal to the aggregate value of all bequests received,

$$\sum_a IV_t^a = 0. \quad (48)$$

Using the balance of inheritances and bequests in (47) yields

$$G A_{t+1} = R_{t+1} (A_t + Y_t + Z_t - C_t), \quad A_t \equiv \sum_a A_t^a. \quad (49)$$

⁵In the empirical implementation, we take per capita profiles for $\bar{A}^a = A^a/N^a$, $\bar{y}^a$, $\bar{C}^a$ and possibly z^a , and use intergenerational transfers iv^a (inheritances and bequests) to reconcile the profiles.

To close the model, we need a precise expression for aggregate labor related income Y . Noting $N^{w,a} = x^a N^a$ and $N^{r,a} = (1 - x^a) N^a$ as well as $\bar{y}_t^a$ in (43) yields

$$Y = \sum_a Y^a, \quad Y^a = \bar{y}_t^a N_t^a = y_t^a \cdot N^{w,a} + (1 - t^{p,a}) (\bar{P}_t^a + p_t^0) \cdot N^{r,a}. \quad (50)$$

To obtain the aggregate consumption function, one must know the aggregate expected foresight variable. Aggregation is simple for all variables that are independent of life-cycle history so that the future looks the same for all agents in a given group. From (23) and (30), human wealth per capita is independent of history, $H_t^a = H_{\alpha,t}^a$, giving aggregate human wealth equal to $H_t^a N_t^a$. The same holds for transfer wealth T_t^a per capita. The main difficulty lies in foresight variables that are tied to historically accumulating stocks such as pension rights. For example, (13) cannot be explicitly aggregated. However, the value of earnings linked pensions is $S_{\alpha,t}^a = \tilde{\lambda}_t^a P_{\alpha,t}^a$ by propositions 2, 4 and 6 where the shadow price $\tilde{\lambda}_t^a$ was shown to be independent of history. The aggregate value can thus be written as $S_t^a N_t^a = \tilde{\lambda}_t^a P_t^a = \tilde{\lambda}_t^a \bar{P}_t^a \cdot N_t^a$ where S_t^a is defined in per capita terms. Unfortunately, this way of writing pension wealth is not possible for young groups who have not yet accumulated positive entitlements because they are not yet included in the earnings link ($m^a = 0$ for those groups, giving $P^a = 0$). Nevertheless, since they expect to accumulate at a later stage of life, they clearly have a positive pension wealth. To circumvent this problem, we replace all heterogeneous terms $P_{\alpha,t}^a$ by their average per capita component $\bar{P}_t^a$ which is independent of history. We thus assure that aggregate values are the same: $\sum_{\alpha} P_{\alpha,t}^a N_{\alpha,t}^a = P_t^a = \sum_{\alpha} \bar{P}_t^a N_{\alpha,t}^a$. Inserting $P_{\alpha,t}^a = \bar{P}_t^a$ in the relevant difference equations, we essentially make $S_{\alpha,t}^a = S_t^a$ independent of history, allowing to solve it directly the same way as H_t^a and T_t^a . Use this way of solving for S_t^a , and note also that the inverse of the marginal propensity to consume Δ_t^a is independent of history. The aggregate consumption function is thus obtained as

$$\begin{aligned} Q_t^a &= \frac{1}{\Delta_t^a} [A_t^a + (H_t^a + T_t^a + S_t^a) N_t^a], \\ S_t^a &= (1 - x_t^a) (1 - t^{p,a}) \bar{P}_t^a - \mu_{w,t}^a \cdot \bar{s}_t^a + \frac{\gamma^a G \bar{S}_{t+1}^a}{R_{t+1} \Omega_{t+1}^a}, \\ \bar{S}_{\alpha,t+1}^a &= \omega^a S_{\alpha,t+1}^a + (1 - \omega^a) (\Lambda_{t+1}^a)^{1-\rho} S_{\alpha',t+1}^{a+1}. \end{aligned} \quad (51)$$

There is no current dividend ($\bar{s}_t^a = 0$) for young groups ($\mu_{w,t}^a = 1$) if they do not yet benefit from the tax benefit link ($m^a = 0$). Nevertheless, the solution S_t^a is strictly positive also for the

youngest groups, reflecting their expectation to become entitled to an earnings linked pension later in life. The forward looking laws of motion of H_t^a and T_t^a were already stated earlier.

The utility value of consumption Q_t^a is lower than actual goods consumption due to the existence of earnings related effort costs. Consumer spending on goods is

$$\begin{aligned} a < a^m : & \quad C_t^a = Q_t^a + \bar{\varphi}_t^a \cdot N_t^a, \\ a = a^m : & \quad C_t^a = Q_t^a + [x_t \cdot \bar{\varphi}_t^a + \varphi_R(x_t)] \cdot N_t^a, \\ a > a^m : & \quad C_t^a = Q_t^a. \end{aligned} \tag{52}$$

Aggregate consumption values are defined as in (49), i.e. $C_t = \sum_a C_t^a$ and $Q_t^a = \sum_a Q_t^a$.

3 Production and the Labor Market

3.1 Job Matching

A fraction $1 - \varepsilon^a$ of people get jobs instantaneously or are always employed. There are $\varepsilon^a x^a N^a$ people looking for jobs where $x^a = x$ for the mixed group and $x^a = 1$ for full workers. Age group a effectively searches $\zeta^a \varepsilon^a x^a N^a$ units, giving a total supply of $L^M = \sum_a \zeta^a \varepsilon^a x^a N^a$ units that determine, together with v vacancies, the number of matches. Market frictions result in M matches so that only a share f of effective searchers and q of vacancies get matched,

$$f \cdot L^M = M(L^M, v) = q \cdot v, \quad L^M \equiv \sum_a \zeta^a \varepsilon^a x^a N^a, \quad \Theta \equiv v/L^M, \tag{53}$$

where Θ denotes labor market tightness. A fraction $f(\Theta)$ of all search units and a fraction $q(\Theta)$ of all vacancies are successfully matched. Assuming $M(\cdot)$ to be quasiconcave and linear homogeneous implies $f'(\Theta) > 0 > q'(\Theta)$ and $q(\Theta) = \Theta f(\Theta)$.⁶

The smallest homogeneous unit counts N_α^a people. A fraction $1 - \varepsilon^a$ is employed instantaneously without search. A fraction ε^a must search and finds a job with probability $\zeta^a f$. Hence, $(1 - \varepsilon^a + \zeta^a f \cdot \varepsilon^a) x^a N_\alpha^a$ people are employed while $(1 - \zeta^a f) \varepsilon^a x^a N_\alpha^a$ remain unemployed. Summing up gives the total mass $x^a N_\alpha^a$. The unemployment rate is identical for all people in group

⁶Use $M = M_0 \cdot (L^M)^\eta v^{1-\eta}$ and get $f = M_0 \Theta^{1-\eta}$, $q = M_0 / \Theta^\eta$ and $f = \Theta \cdot q$.

a and equals $u^a = (1 - \zeta^a f) \varepsilon^a$ as in (14). The fraction of job seekers of type α and a is ϕ_α^a . By the definition of L^M , these shares sum up to unity over all types,

$$\phi_\alpha^a \equiv \zeta^a \varepsilon^a x^a N_\alpha^a / L^M, \quad \phi^a = \sum_\alpha \phi_\alpha^a, \quad \sum_a \sum_\alpha \phi_\alpha^a = \sum_a \phi^a = 1. \quad (54)$$

When a firm announces a vacancy, it finds a worker with probability q and with probability ϕ_α^a that person is of type a, α . From (54), $\zeta^a \varepsilon^a x^a N_\alpha^a = \phi_\alpha^a \cdot L^M$. Multiplying this by f and noting $fL^M = qv$ by (53) yields

$$\zeta^a f \cdot \varepsilon^a x^a N_\alpha^a = \phi_\alpha^a \cdot qv \quad \Rightarrow \quad \zeta^a f \cdot \varepsilon^a x^a N^a = \phi^a \cdot qv. \quad (55)$$

Summing up over α on both sides yields the last equality.

When a firm posts v vacancies, it gets qv workers with different age and skills. Each worker of type α, a has skill θ^a and works l^a hours. Skills θ^a of workers of different age $a \leq a_L$ are assumed perfect substitutes. Total effective manpower is thus

$$\begin{aligned} L_t^D &= \sum_a \sum_\alpha l_t^a \theta^a \cdot [(1 - \varepsilon^a) x_t^a N_{\alpha,t}^a + \phi_{\alpha,t}^a q_t v_t], \\ L_t^D &= \sum_a l_t^a \theta^a \cdot [(1 - \varepsilon^a) x_t^a N_t^a + \phi_t^a q_t v_t] = \sum_a (1 - u_t^a) l_t^a \theta^a x_t^a N_t^a. \end{aligned} \quad (56)$$

The first term in the square bracket relates to the workforce employed without matching. The second term corresponds to (55) and indicates the addition to the workforce by new hiring in the matching market. The second equation results upon using $\sum_\alpha \phi_\alpha^a = \phi^a$ in (54) and reflects symmetry within each age group. The last equality uses $qv = fL^M$ and the definition of ϕ^a as well as $u^a = (1 - \zeta^a f) \varepsilon^a$.

The wage w^a per efficiency unit varies with age. That may be rationalized by increasing bargaining power, seniority considerations or may be induced by non-uniform life-cycle taxes. Summing over $a \leq a_L$ and using the same arguments as before, the wage bill is

$$W_t^D = \sum_a w_t^a l_t^a \theta^a \cdot [(1 - \varepsilon^a) x_t^a N_t^a + \phi_t^a q_t v_t]. \quad (57)$$

3.2 Job Creation of Firms

Firms produce output F using capital K and L^D efficiency units of labor. They invest to accumulate capital and hire labor on a search labor market. Investors value the firm because of

its stream of dividends. After financing investment with retained earnings, the firm is able to pay dividends χ . Installing I^K units of new equipment causes installation costs J , measured in terms of foregone output. The total investment cost $I + J$ thus consists of market spending and internal adjustment costs. Capital depreciates at constant rate $\delta^K - 1$. Maintaining a vacancy requires resource costs of κ per job offer. Firms are also assumed to incur fixed costs v_0 which reflect the human resources spent on inframarginal workers who are employed without matching frictions. Technically, these fixed costs eliminate the large rents on inframarginal labor that would arise otherwise, thus inflating firm values and required aggregate savings,

$$\begin{aligned} (a) \quad & F_t \equiv F(K_t, L_t^D), \quad J_t = J(I_t^K, K_t), \\ (b) \quad & \chi_t = F_t - J_t - (v_0 + \kappa v_t) - I_t^K - W_t^D, \\ (c) \quad & GK_{t+1} = I_t^K + \delta^K K_t. \end{aligned} \tag{58}$$

A no-arbitrage condition gives the investors' valuation of the firm's dividend stream. Denoting end of period firm value by V_t , the Bellmann equation of firm maximization is

$$V(K_t) = \max_{v_t, I_t^K} \chi_t + GV(K_{t+1})/R_{t+1} \quad s.t. \quad (56-58). \tag{59}$$

Denoting the shadow price of capital by $\lambda_t^K \equiv \partial V_t / \partial K_t$, optimality and envelope conditions are

$$\begin{aligned} (I_t^K) \quad & \lambda_{t+1}^K / R_{t+1} = 1 + J_{I,t}, \\ (v_t) \quad & \kappa = q_t \cdot \sum_a (F_L - w_t^a) l_t^a \theta^a \phi_t^a, \\ (K_t) \quad & \lambda_t^K = F_K - J_K + \delta^K \lambda_{t+1}^K / R_{t+1}. \end{aligned} \tag{60}$$

The optimality condition for vacancies is easily interpreted. Marginal cost of a vacancy κ is equal to expected marginal benefit from finding a worker. With probability q the firm locates a worker, and with probability $q\phi^a$, this worker is of type a , working l^a hours with skills θ^a . The firm's job surplus per efficiency unit $l^a\theta^a$ is $F_L - w^a$. Hence, the marginal expected job surplus created by an extra vacancy is equal to its marginal resource cost.

Proposition 8 (Hayashi) *The value of the firm is $V_t = \lambda_t^K K_t + V_t^E$, where*

$$V_t^E \equiv \sum_a (F_L - w_t^a) \theta^a l_t^a (1 - \varepsilon^a) x_t^a N^a - v_0 + \frac{GV_{t+1}^E}{R_{t+1}} \tag{61}$$

is the present value of the job rents on instantaneously employed workers.

Proof. Multiply the envelope condition in (60) by K and use the optimality condition for I^K as well as linear homogeneity of $F(\cdot)$ and $J(\cdot)$ and the law of motion for K to get

$$\lambda^K K = F - J - L^D F_L + G K_{t+1} \lambda_{t+1}^K / R_{t+1}. \quad (\text{i})$$

Multiply the vacancy condition by v , rewrite, and use (56-57),

$$\kappa v_t = (L_t^D - \sum_a \theta^a l_t^a (1 - \varepsilon^a) x_t^a N_t^a) F_L - (W_t^D - \sum_a w_t^a \theta^a l_t^a (1 - \varepsilon^a) x_t^a N_t^a).$$

Substitute this into (i) and note (58.b) to get

$$\lambda^K K = \chi_t + G K_{t+1} \lambda_{t+1}^K / R_{t+1} + v_0 - \sum_a (F_L - w_t^a) \theta^a l_t^a (1 - \varepsilon^a) x_t^a N_t^a.$$

Replace the last term using (61) and rearrange,

$$(\lambda_t^K K_t + V_t^E) = \chi_t + \frac{G (\lambda_{t+1}^K K_{t+1} + V_{t+1}^E)}{R_{t+1}}. \quad (\text{ii})$$

This proves the Hayashi's result since the forward looking solution of $(\lambda_t^K K_t + V_t^E)$ in (ii) is the same as that for V_t in (59). ■

Normalizing adjustment costs to zero in a stationary state ($J = J_I = J_K = 0$), we obtain from (60) a particularly simple condition for optimal capital accumulation in the long-run, $F_K = R + \delta^K$, where $R + \delta^K$ is equal to the sum of interest and depreciation rate. In a small open economy, the capital labor ratio is entirely determined by the world interest rate.

3.3 Wage Negotiations

Once a match has been generated, the worker and firm must agree on a wage to start the relationship. Each side gains a surplus that depends on the agreed wage and its fallback position. An agent wants to accept a match when her surplus is positive. A higher wage raises the worker's surplus but reduces the gains to the firm. The worker's surplus is equal to $-dV_{\alpha,t}^a/du_t^a$ because this shows how welfare increases by giving up unemployment benefits in exchange for salary income. Taking the mixed age group, we eventually get

$$\begin{aligned} -\frac{dV_{\alpha,t}^a}{du_t^a} &= (V_{\alpha,t}^a/Q_{\alpha,t}^a)^{1-\rho} x_t^a \theta^a [\Gamma_t^a w_t^a l_t^a - \varphi_L(l_t^a) - b_0^a], \\ \Gamma_t^a &\equiv (1 - b^U) (1 - t^{w,a} - t^{s,a}) + m^a \cdot (1 - b^1) \frac{\mu_t^a}{x^a} \frac{R_{t+1}^P}{R_{t+1}} \frac{\gamma^a \bar{\lambda}_{t+1}^a}{\bar{\eta}_{t+1}^a}. \end{aligned} \quad (62)$$

The square bracket corresponds to the gains to search as captured by the same square bracket in (28). After having located a job (a priori this happens with probability $\zeta^a f$), a worker can obtain this extra income by accepting the job offer. The surplus also reflects the increase in future pensions since the agent accumulates additional entitlements when she earns more wage income today. The factor Γ_t^a shows how the surplus from accepting employment changes when the job pays one Euro more gross income $w^a l^a$, taking account of the adjustment of taxes and benefits that are related to gross income.

The firm's surplus from employing a new worker with skill θ^a is $(F_L - w^a) l^a \theta^a$, see (60). If no wage is agreed and employment is foregone, search costs would be wasted. The firm could not reap this surplus and add it to its total profit. In other words, the firm's fallback position is zero. Wage bargaining maximizes the Nash product,

$$\max_{w^a} (w^a - w_R^a)^\xi (F_L - w^a)^{1-\xi}, \quad w_R^a \equiv (\varphi_L^a + b_0^a) / (l^a \Gamma^a). \quad (63)$$

We ignore constant terms as they do not influence the solution.⁷ We thus consider only the square bracket in (62). In taking out the factor $l^a \Gamma^a$, we define the worker's reservation wage w_R^a per hour and gross of tax. The f.o.c. yields the wage per hour and age group:

$$w_t^a = \xi \cdot F_L + (1 - \xi) \cdot w_{R,t}^a. \quad (64)$$

The reservation wage is the minimum gross wage per hour that makes the worker indifferent between accepting or rejecting the match. If the gross wage were lower, she would reject the job offer and the firm could not get any job rent. A gross wage in excess of this leaves a strictly positive surplus, making the worker keen to accept. However, a higher wage reduces the firm's surplus and thereby discourages job creation.

If the worker is employed, he generates an extra output F_L per hour of work. Bargaining yields a gross hourly wage per efficiency unit that is an average of the worker's marginal product per hour and her reservation wage. Adding up the two round brackets in (63) gives the joint surplus of the match equal to $F_L - w_R^a$. As usual, each side obtains a share of the joint surplus that corresponds to its bargaining power. Rearranging (64) we find that the worker's job rent is $w^a - w_R^a = \xi \cdot (F_L - w_R^a)$, while the firm's job rent is $F_L - w^a = (1 - \xi) \cdot (F_L - w_R^a)$.

⁷This is most easily seen by solving $\max_w [a_1 (w - b)]^\xi [a_2 (F_L - w)]^{1-\xi}$ which yields $w = \xi F_L + (1 - \xi) b$ independent of a_1 and a_2 .

4 General Equilibrium

4.1 Aggregate Wage Income

In proving Walras' Law, consider first asset accumulation (49), joint with (48),

$$GA_{t+1} = R_{t+1} (A_t + Y_t + Z_t - C_t + \zeta_t^B), \quad \zeta_t^B \equiv \sum_a IV_t^a = 0, \quad (65)$$

where the “excess demand” ζ^B is zero in equilibrium.

Aggregate variables such as $A = \sum_a A^a$ sum up over all age groups. To obtain $Y = \sum_a Y^a$, we first rewrite (42-43) as

$$\begin{aligned} Y_t^a &= x_t^a \cdot [(1 - u_t^a) (1 - t^{w,a} - t^{s,a}) w_t^a l_t^a + u_t^a b_t^a] \theta^a N_t^a \\ &: + (1 - x_t^a) \cdot (1 - t^{p,a}) (P_t^a + p_t^0 N_t^a) = \bar{y}_t^a N_t^a, \\ b_t^a &= b^U \cdot (1 - t^{w,a} - t^{s,a}) w_t^a l_t^a + b_0^a. \end{aligned} \quad (66)$$

Observe that x^a is either 1, x or 0 depending on the age group. This definition can be cross-checked with (50).

4.2 Fiscal Budget Constraints

We have separate budget constraints for general government and the pension system. The pension system collects contributions and spends on pensions. It may also receive a transfer Z_t^P from general government:

$$Z_t^P = \sum_a [(1 - x_t^a) (P_t^a + p_t^0 N_t^a) - t^{s,a} (1 - u_t^a) l_t^a w_t^a \theta^a x_t^a N_t^a], \quad (67)$$

where one must take care of the definition of x^a . The excess of spending on pensions over contribution revenues is covered by transfers from the general budget.

The general government collects a wage tax to pay for unemployment benefits, other transfers to households, public consumption C^G and public debt service D^G . Tax revenues and spending for unemployment benefits and other transfers are

$$\begin{aligned} T_t^G &= \sum_a [t^{w,a} (1 - u_t^a) l_t^a w_t^a \theta^a x_t^a N_t^a + t^{p,a} (1 - x_t^a) (P_t^a + p_t^0 N_t^a)], \\ B_t &= \sum_a u_t^a b_t^a \theta^a x_t^a N_t^a, \quad Z_t = \sum_a z_t^a N_t^a. \end{aligned} \quad (68)$$

The government budget constraint thus amounts to

$$GD_{t+1}^G = R_{t+1} [D_t^G + C_t^G + Z_t^P + Z_t + B_t - T_t^G]. \quad (69)$$

4.3 Walras' Law

As a first step, we consolidate disposable wage related incomes of households with wage related taxes and benefits of the public sector. Using (67-68), we find that household disposable wage income Y as defined in (66) is

$$Y_t = \sum_a w_t^a l_t^a \theta^a (1 - u_t^a) x_t^a N_t^a - T_t^G + Z_t^P + B_t. \quad (70)$$

Using $1 - u_t^a = 1 - \varepsilon^a + \zeta^a f \varepsilon^a$ and comparing with W^D in (57), the first term can be written as

$$\begin{aligned} \sum_a w_t^a \theta^a l_t^a (1 - u_t^a) x_t^a N_t^a &= W_t^D + \sum_a w_t^a \theta^a l_t^a [f_t \cdot \zeta_t^a \varepsilon^a x_t^a N_t^a - \phi_t^a q_t v_t] \\ &= W_t^D + (f_t L_t^M - q_t v_t) \cdot \sum_a w_t^a \theta^a l_t^a \phi_t^a. \end{aligned} \quad (71)$$

The last equation uses $\zeta^a \varepsilon^a x^a N^a \equiv \phi^a L^M$ and extracts ϕ^a from the bracket. Substituting in (70) yields

$$Y_t = W_t^D - T_t^G + Z_t^P + B_t + \zeta_t^L, \quad \zeta_t^L \equiv (f_t L_t^M - q_t v_t) \cdot \sum_a w_t^a \theta^a l_t^a \phi_t^a. \quad (72)$$

The bracket can be viewed as an excess demand for matches, and the multiplicative term is understood as an average of wage income $w^a \theta^a l^a$ per match, weighted by ϕ^a . Hence, the last term is the valued excess demand for matches.

Capital market equilibrium requires that household assets are invested in domestic equity and domestically and internationally traded bonds,

$$A_t = V_t^K + D_t^F + D_t^G. \quad (73)$$

Proposition 9 *The current account follows from Walras' Law:*

$$GD_{t+1}^F = R_{t+1} (D_t^F + T_t^B), \quad TB_t = F_t - v_0 - \kappa v_t - J_t - I_t^K - C_t^G - C_t. \quad (74)$$

Proof. Define the following excess demands (V refers to firm value),

$$\zeta_t^A \equiv A_t - V_t - D_t^F - D_t^G, \quad (\text{i})$$

$$\zeta_t^G = D_t^G + C_t^G + Z_t^P + Z_t + B_t - T_t^G - GD_{t+1}^G/R_{t+1}, \quad (\text{ii})$$

$$\zeta_t^F = D_t^F + TB_t - GD_{t+1}^F/R_{t+1}, \quad (\text{iii})$$

$$\zeta_t^L = (f_t L_t^M - q_t v_t) \sum_a w_t^a \theta^a l_t^a \phi_t^a = (f_t - \Theta_t q_t) \sum_a w_t^a \theta^a l_t^a \phi_t^a L_t^M, \quad (\text{iv})$$

$$\zeta_t^B = \sum_a IV_t^a. \quad (\text{v})$$

Use this in (65) and substitute (i), (59) for V and, as a last step, (58) for χ . Using (ii) for the government budget as well as disposable labor income Y in (72), and noting the definition of TB_t , we have

$$(\zeta_{t+1}^A + D_{t+1}^F) G/R_{t+1} = \zeta_t^A + \zeta_t^L + \zeta_t^G + \zeta_t^B + D_t^F + TB_t. \quad (\text{vi})$$

Using (iii) yields

$$\zeta_t^L + \zeta_t^G + \zeta_t^B + \zeta_t^F + \zeta_t^A - \zeta_{t+1}^A G/R_{t+1} = 0. \quad (\text{vii})$$

The portfolio condition holds identically in t , i.e. $\zeta_t^A = 0$. Only in $t+1$, savings and investment implies $\zeta_{t+1}^A \neq 0$ out of equilibrium. In equilibrium, $\zeta_t^L = \zeta_t^G = \zeta_t^B = \zeta_t^A = \zeta_{t+1}^A = 0$, implying $\zeta_t^F = 0$ by (vii). This proves (74). ■

Proposition 10 *The savings investment identity (in primary balances) holds:*

$$\begin{aligned} TB_t &= S_t^H + S_t^G, \\ S_t^H &= \chi_t + W_t^D - T_t^G + Z_t^P + Z_t + B_t - C_t, \\ S_t^G &= T_t^G - (C_t^G + Z_t^P + Z_t + B_t). \end{aligned} \quad (\text{75})$$

Proof. We start with (65) and replace Y by (72),

$$GA_{t+1}/R_{t+1} - A_t - \zeta_t^L - \zeta_t^B = W_t^D - T_t^G + Z_t^P + Z_t + B_t - C_t. \quad (\text{i})$$

The l.h.s. is the increase in financial wealth. Since all business investment is financed with retained profits, there are no equity issues. The increase in V is purely capital gains and not measured savings. Replace assets by the portfolio identity (74.i), impose no-arbitrage as in (59),

$V_t = \chi_t + GV_{t+1}/R_{t+1}$, and note measured savings (new assets actually bought by households),

$$\begin{aligned} S_t^H &= (D_{t+1}^F + D_{t+1}^G + \zeta_{t+1}^A) G/R_{t+1} - D_t^F - D_t^G - \zeta_t^A - \zeta_t^L - \zeta_t^B, \\ S_t^G &= D_t^G - GD_{t+1}^G/R_{t+1} - \zeta_t^G, \\ TB_t &= GD_{t+1}^F/R_{t+1} - D_t^F + \zeta_t^F. \end{aligned} \quad (\text{ii})$$

The second and third equations are taken from (74.ii-iii). The flow terms exactly correspond to (75). From (ii), we have

$$S_t^H + S_t^G - TB_t = \zeta_{t+1}^A G/R_{t+1} - \zeta_t^A - \zeta_t^L - \zeta_t^G - \zeta_t^F - \zeta_t^B = 0. \quad (\text{iii})$$

The proof is complete since (iii) holds identically by Walras' Law, see (74.vii). To check, compute from (75) $S_t^H + S_t^G - TB_t = 0$ by inserting the definitions of χ and TB . ■

5 Welfare Analysis

5.1 Equivalent Variations

Our welfare computations are based on equivalent variations which give the change in income or wealth that is equivalent to the change in utility if compared at the same prices. Equations (i-ii) of Propositions 1,3 and 5 yield indirect life-time utility $V_{\alpha,t}^a$ of an agent in age group a with history α . We cannot observe individual welfare separately for each possible life-cycle history. But we can compute separately the welfare per capita of a newborn (since they are not yet heterogeneous in their history) and the welfare of an average existing agent in age group a . We strictly express all values per capita, and use $\sum_{\alpha \in \mathcal{N}_t^a} A_{\alpha,t}^a N_{\alpha,t}^a = A_t^a \equiv \bar{A}_t^a N_t^a$ to get assets $\bar{A}_t^a$ per capita. Upon aggregation of (23.a-b), per capita welfare in group a is

$$V_t^a = P_t^a \cdot \mathcal{W}_t^a, \quad P_t^a \equiv (\Delta_t^a)^{1/(\sigma-1)}, \quad \mathcal{W}_t^a \equiv \bar{A}_t^a + H_t^a + S_t^a + T_t^a. \quad (76)$$

Welfare of a newborn in age group 1 would also be the same, except that wealth per capita would be $\mathcal{W}_t^a \equiv H_t^a + S_t^a + T_t^a$ only since newborns are not yet endowed with any assets. Aggregate welfare of group a would be simply $V_t^a N_t^a$. Inverting indirect utility yields an intertemporal expenditure function which gives the level of life-time wealth per capita necessary to obtain utility V_t^a at intertemporal prices P_t^a ,

$$\varrho(P_t^a, V_t^a) = V_t^a / P_t^a. \quad (77)$$

When considering the welfare impact of a policy shock, we refer to equilibrium values *before* a policy shock by an upper index of zero and leave away the index if we refer to values after the shock, $P_t^{0,a}$ and P_t^a , for example. Taking initial prices as a reference, the equivalent variation measure EV gives the wealth equivalent change in welfare per capita. It is defined (separately for each generation or population group) as

$$EV_t^a = \varrho \left(P_t^{0,a}, V_t^a \right) - \mathcal{W}_t^{0,a} = V_t^a / P_t^{0,a} - \mathcal{W}_t^{0,a}. \quad (78)$$

5.2 Aggregate Welfare Measure

Suppose the policy change became effective at date $t = 1$. To characterize redistribution, we distinguish between present and future generations. Existing present generations were born in periods $t \leq 1$. We report the aggregate welfare impact for each group living in $t = 1$. The equivalent variation may be expressed as a percentage of life-time wealth $\mathcal{W}^{0,a}$.

We separately report the results for all new, future generations entering in periods $t > 1$. Newborns $N_{t,t}^1$ are always a fraction of the first age group with total size N_t^1 . New agents start life without assets but have the same human and social security wealth per capita, H^a , S^a and T^a , as existing members of group 1. Life-time wealth of a new born is, thus, $\mathcal{W}_{t,t}^1 = H_t^1 + S_t^1 + T_t^1$, leading to an equivalent variation $EV_{t,t}^1$ per capita as in (78).

The aggregate equivalent welfare change of new generations born in period 2 or later is

$$EV_1^N = \sum_{t=2}^{T-1} EV_{t,t}^1 \cdot N_{t,t}^1 \prod_{u=2}^t \frac{G}{R_u} + \frac{R}{r-g} EV_{T,T}^1 \cdot N_{T,T}^1 \prod_{u=2}^T \frac{G}{R_u}. \quad (79)$$

The first term adds up the variations of new generations born until date T . The second term collects all those born at date T or later where T is the length of the transition period. At that date, the economy has already approached a steady state, giving constant values thereafter. The last multiplicative term discounts the welfare gains of all new generations born in periods T or later to the beginning of period 1.

Adding welfare changes of new and old generations, we arrive at an aggregate measure,

$$EV_1 = EV_1^N + EV_1^O, \quad EV_1^O = \sum_a EV_1^a \cdot N_1^a. \quad (80)$$

The aggregate welfare change is equivalent to an amount of wealth EV_1 , as measured at the beginning of period 1. To express welfare gains relative to GDP, we convert this wealth

measure into permanent income, i.e. we compute a permanent annuity with the same present value, taking the initial interest rate as the discount rate,

$$EV_1 = ev + ev \frac{G}{R} + ev \frac{G}{R^2} + \dots = ev \frac{R}{r-g} \Rightarrow ev = \frac{r-g}{R} EV_1. \quad (81)$$

The aggregate welfare change in percent of GDP is then $100 \times ev/GDP^0$.

6 Calibration

6.1 Retirement

According to Gruber and Wise (2005) and Boersch-Supan (2000), the retirement decision is arguably the most important behavioral response to pension reform. The retirement elasticity is taken from empirical studies. Börsch-Supan (2000) estimates that a decrease in benefits by 12% would reduce the retirement probability of the 60 years old from 39.3% to 28.1%. This amounts to a semi-elasticity of the retirement decision of $\varepsilon^R = 1$. Given the similarity of pension systems, we take this semi-elasticity for Germany also as representative for Austria. We can use this estimate to calibrate the retirement cost function which we specify as

$$\varphi_R(x) = \varphi_1 \cdot \varepsilon_x \exp[x/\varepsilon_x] - \varphi_0. \quad (82)$$

The retirement condition in (33-34) means that people in the mixed age group postpone retirement, i.e. the participation rate x increases, if work becomes more attractive relative to retirement. Suppressing the group index a and denoting the semi-elasticity of retirement by ε_R , the differential of the retirement condition $\varphi'_R(x) = (1 - t^R)(1 - u)wl\theta - \bar{\varphi}$ yields

$$dx = -\varepsilon_R \cdot \frac{d[(1 - t^p)(\bar{P} + p^0)]}{(1 - t^p)(\bar{P} + p^0)}, \quad \varepsilon_R \equiv \frac{(1 - t^p)(\bar{P} + p^0)}{\varphi'_R(x)} = \varepsilon_x \cdot \frac{(1 - t^p)(\bar{P} + p^0)}{(1 - t^R)(1 - u)wl\theta - \bar{\varphi}}.$$

The second equality in ε_R expands by $\varphi'_R(x) = (1 - t^R)(1 - u)wl\theta - \bar{\varphi}$ and uses $\varphi'_R/\varphi''_R = \varepsilon_x$. The econometric estimate of the elasticity ε_R is used to infer the parameter

$$\varepsilon_x = \varepsilon_R \cdot \frac{(1 - t^R)(1 - u)wl\theta - \bar{\varphi}}{(1 - t^p)(\bar{P} + p^0)}. \quad (83)$$

The fraction is known from data and the normalization $\bar{\varphi} = 0$ of utility costs in the initial equilibrium. Having ε_x , the f.o.c. gives $\varphi_1 = [(1 - t^R)(1 - u)wl\theta - \bar{\varphi}] / \exp[x/\varepsilon_x]$ while φ_0 is chosen to normalize the retirement utility cost to zero initially, $\varphi_0 = \varphi_1 \cdot \varepsilon_x \exp[x/\varepsilon_x] - \varphi_R(x)$.

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