An Extended Stein's Lemma for Asset Pricing

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First draft: May 2007; Current draft: 8 August 2008

Forthcoming in *Applied Financial Economics Letters*

Abstract

Stein's lemma is extended to the case where asset returns have skewed and leptokurtic distributions. The risk premium is still the negative of the covariance of the excess return with the log SDF. The riskneutral distribution has a simple form, but it is a non-trivial transformation of the physical distribution.

Keywords: Risk premium, stochastic discount factor, change of measure

JEL Classification Numbers: G12

Let Z be the excess return on an asset and M be a stochastic discount factor (SDF). Most asset pricing models then imply that

$$\begin{aligned} E(ZM) &= 0, \text{ so} \\ E(Z) &= -\text{Cov}(Z, M)/E(M), \end{aligned} \tag{1}$$

where $E()$ denotes the expectations and $\text{Cov}()$ the covariance (possibly conditional).

In many specifications, this expression is inconvenient since the SDF is a non-linear function. For instance, with a constant relative risk aversion (CRRA) utility function, $C^{1-\gamma}/(1-\gamma)$, the SDF derived from optimal consumption/investment is proportional to $(C_t/C_{t-1})^{-\gamma}$. This non-linearity complicates both the interpretation and the estimation/evaluation of the model.

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A commonly applied simplification is to assume that Z_i and $\ln M$ have a bivariate normal distribution. Then, Stein's lemma¹ gives

$$E(Z) = -\text{Cov}(Z, \ln M). \quad (2)$$

This is a great simplification when it is reasonable to assume that $\ln M$ is normally distributed—which it might well be when the log SDF is linear in macro variables like consumption growth, labour supply and inflation (for a nominal SDF).² For instance, with the CRRA utility function the log SDF equals $-\gamma$ times the growth rate of consumption.

However, this result relies on the strong assumption that returns are also normally distributed—which they typically are not. For instance, most equity returns appear to be both skewed and have fat tails. For that reason, the Proposition 1 below extends Stein's lemma to the case where the log SDF is still normally distributed, but where the asset return is allowed to have a mixture normal (and where correlations differ between the normal components). Clearly, this setup could be given a structural interpretation in terms of a regime switching model, but this is (equally clearly) not necessary.

Figure 1 provides a simple illustration (based on a mixture of two normal distributions) where the excess return features a really long left tail.

Proposition 1 *Assume (a) the joint distribution of x and y is a mixture of n bivariate normal distributions; (b) the mean and variance of y is the same in each of the n components; (c) $h(y)$ is a differentiable function such that $E[|h'(y)|] < \infty$. Then $\text{Cov}[x, h(y)] = E[h'(y)] \text{Cov}(x, y)$. In addition, $\text{Cov}(x, y) = \sum_{i=1}^n \alpha_i \text{Cov}_i(x, y)$, where α_i and $\text{Cov}_i(x, y)$ are the weight of and the covariance in the i th mixture component respectively. (A straightforward proof is in the appendix.)*

The following empirical example illustrates the potential of this framework to describe historical data on returns and macro variables.

Empirical Example *Quarterly data (1952–2005) for US stock market excess returns (from French (2001)) and US consumption growth (real, per capita, nondurable goods*

¹Stein's lemma says that if x and y have a bivariate normal distribution and $h(y)$ is a differentiable function such that $E[|h'(y)|] < \infty$, then $\text{Cov}[x, h(y)] = \text{Cov}(x, y) E[h'(y)]$. Here we have $x = Z$, $y = \ln M$, and $h(y) = \exp(y)$ so $h'(y) = M$. Therefore, $\text{Cov}(Z_i, M) = \text{Cov}(Z_i, \ln M) E(M)$. See, for instance, Cochrane (2001).

²More generally, for an asset with random payoff X , the current price is $P = E(X) E(M) + \text{Cov}(X, \ln M) E(M)$.

Figure 1: Example of a bivariate mixed-normal distribution. The marginal distributions are drawn at the back (scaled, to fit the figure). The figure uses a mixture of 2 normal distributions, with weights 0.7 and 0.3 respectively. In the first component, the means and variances are 1.3 and 1 for Z , 0 and 1 for $\ln M$ and the covariance is -0.1. In the second component, the means and variances are -1.9 and 0.3 for Z , 0 and 1 for $\ln M$ and the covariance is -0.9.

and service, from NIPA) is used in a maximum likelihood estimation. Consumption growth is assumed to have the same mean and standard deviation in the two gaussian components (estimated to be 0.51% and 0.47% respectively). Stock returns have the following estimated means and standard deviations: in component 1 (1.4,8.2); in component 2 (2.1,4.4) with a weight of 0.62 on the second component. The estimated correlations between consumption growth and returns are 0.40 and 0.18 in the two components respectively. Component 1 can therefore be described as a low return/high volatility state where returns and consumption are fairly strongly correlated. Component 2 is the opposite.

The mixture distribution allows not only a flexible modelling of the marginal distribution of excess returns, but also of risk compensation. Proposition 1 implies that we can write the risk premium in (2) as

$$\sum_{i=1}^n \alpha_i E_i(Z) = -\sum_{i=1}^n \alpha_i \text{Cov}_i(Z, \ln M), \quad (3)$$

where α_i , $E_i(Z)$, and $\text{Cov}_i(Z, \ln M)$ are the weight of, mean excess return and covariance in the i th component distribution, respectively. This is a convenient form, but still

allows considerable modelling freedom, especially since nothing requires us to match the mean excess return and covariance in each component. As a consequence, the mapping (change of measure) from the physical distribution and the riskneutral distribution (and back) is also convenient, but is far from trivial. The exact mapping is demonstrated in Proposition 2 below.

Proposition 2 *The riskneutral probability density function equals the physical density function times $E(M|Z)/E(M)$. Under the assumptions in Proposition 1, the physical density is $\sum_{i=1}^n \alpha_i \phi[Z; E_i(Z), \text{Var}_i(Z)]$, where $\phi()$ is a gaussian density with the given mean and variance. The riskneutral density is the same, except that the means are $E_i(Z) + \text{Cov}_i(Z, \ln M)$ instead of just $E_i(Z)$. (A straightforward proof is in the appendix.)*

See Figure 2 for an illustration of the physical and riskneutral distributions, based on the parameters in Figure 1. (Recall that a riskneutral distribution is such that the price of any derivative equals the discounted expected value of the payoff.) The figure illustrates how the riskneutral distribution has a lower mean (zero) than the physical distribution: this is a risky asset since the covariance with the SDF is negative. It is also clear that this change of measure entails more than just a shift of the mean: the lower tail is shifted relatively more, which alters the shape of the distribution—and increases the variance. As a consequence, the ratio of the two density functions (the Radon-Nikodym derivative) is non-monotonic. These results stand in sharp contrast to the case when the excess return is normally (or lognormally, for that matter) distributed. In that case, the riskneutral distribution is just a parallel shift of the entire distribution, which leaves the variance unchanged and the ratio of the two densities is monotonic.

A Appendix

A.1 Proof of Proposition 1

Let $\phi(z; \mu_i, \Sigma_i)$ be a bivariate normal pdf (evaluated at z) with the mean vector μ_i and covariance matrix Σ_i . Let the vector $z = (x, y)$ have a mixture of n bivariate normal distributions, $\sum_{i=1}^n \alpha_i \phi(z; \mu_i, \Sigma_i)$ where $\sum_{i=1}^n \alpha_i = 1$ and

$$\mu_i = \begin{bmatrix} E_i(x) \\ E_i(y) \end{bmatrix} \text{ and } \Sigma_i = \begin{bmatrix} \text{Var}_i(x) & \text{Cov}_i(x, y) \\ \text{Cov}_i(x, y) & \text{Var}_i(y) \end{bmatrix}.$$

Figure 2: Example of Physical and riskneutral distribution. The figure uses the same parameters as in Figure 1.

Let $h(y)$ be a differentiable function such that $E_i |h'(y)| < \infty$. Calculating $\text{Cov}[x, h(y)] = E[xh(y)] - E(x)E[h(y)]$ is straightforward since $E[xh(y)] = \sum_{i=1}^n \alpha_i E_i[xh(y)]$. This gives

$$\text{Cov}[x, h(y)] = \sum_{i=1}^n \alpha_i \{ \text{Cov}_i[x, h(y)] + E_i(x)E_i[h(y)] \} - E(x)E[h(y)].$$

Assume that $E_i(y) = E(y)$ and $\text{Var}_i(y) = \text{Var}(y)$, so the marginal distribution of y is a normal distribution. This implies that $E_i[h(y)] = E[h(y)]$, so the covariance simplifies to

$$\text{Cov}[x, h(y)] = \sum_{i=1}^n \alpha_i \text{Cov}_i[x, h(y)]. \quad (*)$$

Stein's lemma for each mixture component says $\text{Cov}_i[x, h(y)] = \text{Cov}_i(x, y)E_i[h'(y)]$. Using this in (*) together with $E_i[h'(y)] = E[h'(y)]$ gives

$$\begin{aligned} \text{Cov}[x, h(y)] &= E[h'(y)] \sum_{i=1}^n \alpha_i \text{Cov}_i(x, y) \\ &= E[h'(y)] \text{Cov}(x, y), \end{aligned}$$

where the second equality follows from applying the result in (*) on the identity function.

A.2 Proof of Proposition 2

For the sake of generality, this proof is made in terms of an asset with a price that may differ from zero, so the payoff S is not necessarily an excess return. Otherwise, S has the same distributional properties as the excess return in the propositions.

Consider pricing a derivative $g(S)$ of an underlying payoff S . The riskneutral distri-

bution (denoted by a star*) is such that the price equals the price of a bill (denoted B) times the expected payoff, $B E^* g(S)$. At the same time, the price must satisfy $E M g(S)$. Equating these expressions and writing in terms of the integrals and density functions (generically denoted $f()$) give

$$B \int_S f^*(S) g(S) dS = \int_S E(M|S) f(S) g(S) dS.$$

The only way to make this hold for any derivative is by letting $B f^*(S) = E(M|S) f(S)$. We also know that $B = E M$. This demonstrates that $f^*(S) = f(S) E(M|S) / E M$.

To prove the second part of the proposition, notice that the distribution of $\ln M$ conditional on S can be written

$$\frac{f(S, \ln M)}{f(S)} = \sum_{i=1}^n \alpha_i \frac{f_i(S)}{f(S)} f_i(\ln M|S),$$

where $f_i()$ denotes the i th component of the density function. The conditional expectation is therefore

$$E(M|S) = \sum_{i=1}^n \alpha_i \frac{f_i(S)}{f(S)} E_i(M|S), \quad (a)$$

where $E_i(M|S)$ is the conditional expectation in the i th component.

In the i th component, Z and $\ln M$ are normally distributed, so the conditional distribution is straightforward to calculate (the mean is $E_i \ln M + (S - E_i S) \text{Cov}_i(S, \ln M) / \text{Var}_i(S)$ and variance $\text{Var}_i(\ln M) - \text{Cov}_i(S, \ln M)^2 / \text{Var}_i(S)$). We then have

$E_i(M|S) = E(M) \exp(\Upsilon_i)$, where

$$\Upsilon_i = (S - E_i S) \text{Cov}_i(S, \ln M) / \text{Var}_i(S) - \text{Cov}_i(S, \ln M)^2 / \text{Var}_i(S) / 2, \quad (b)$$

where we have used the fact that $E(M) = \exp[E_i \ln M + \text{Var}_i(\ln M)/2]$.

Combining $f^*(S) = f(S) E(M|S) / E(M)$ with (a) and (b) gives

$$f^*(S) = \sum_{i=1}^n \alpha_i f_i(S) \exp(\Upsilon_i).$$

Simple algebra then gives that $f_i(S) \exp(\Upsilon_i)$ is a gaussian density function with mean $E_i S + \text{Cov}_i(S, \ln M)$ and variance $\text{Var}_i(S)$.

References

- Cochrane, J. H., 2001, *Asset Pricing*, Princeton University Press, Princeton, New Jersey.
- French, K. R., 2001, "U.S. Research Returns Data," <http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/>.