Asset Allocation, Longevity Risk, Annuitisation and Bequests

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Outline

- Motivation
- Preliminaries
- Optimisation Problem
- Results
- Conclusions
Importance of the End of the Life-Cycle

- Rising Conditional Life Expectancies
- Growing Number of DC Plans
- Continuing Wealth Concentration Among Pensioners
- Input for Labour Models
Technical View of the Pensioner’s Problem

- Consumption/Portfolio Optimisation \((c, \pi)\)  
  → Financial Market Risk

- Optimal Annuitisation Decision \((\tau)\)  
  → Longevity Risk

⇒ Combined Optimal Stopping and Optimal Control Problem (COSOCP)
Literature Overview

- Classical Literature
  - Merton (1969) → Stochastic Control
  - Vast Literature Imposing a Fixed or Infinite Planning Horizon
  - Yaari (1965) → Uncertain Lifetime
  - Richard (1975) → Reversible Annuities

- Few Normative Models with Irreversible Annuities and Uncertain Lifetime, i.e.
  - Stabile (2006):
    Parsimonious Continuous-Time Model Annuitisation Rule as a Controlled Stopping Time
  - Milevsky and Young (2007):
    Focus on Mortality Law and Annuitisation Schemes

⇒ Both Papers: No Bequest Motive
Extensions to the Model of Stabile (2006)

- Inclusion of a Bequest Motive
- Prior Life Insurance and Subsistence Level of Bequest
- Economically Relevant Risk Aversion ($\gamma > 1$)
- New Solution Method with Duality Arguments
Main Model Assumptions

- Utility Maximisation (Consumption, Annuity, Bequest; Identical Relative Risk Aversion)
- No Stochastic Income → No Labour Income
- Prior Decision on Annuitisation and Life Insurance Taken as Given
- Annuitisation of Entire Wealth and Consumption of Entire Annuity → All-or-Nothing Framework
- Irreversible Annuitisation Decision
- One Riskless Asset, One Risky Asset (Geometric Brownian Motion)
- Exponential Mortality Law
Model Basics

- **Life Expectancy and Annuity:**

\[
E^S[T_x] = \int_0^\infty (tp_x^S) \, dt = \int_0^\infty e^{-\lambda_x^S t} \, dt = \frac{1}{\lambda_x^S} \quad \text{and} \quad E^O[T_x] = \frac{1}{\lambda_x^O}
\]

\[
\bar{a}_x = \int_0^\infty e^{-rt} (tp_x^O) \, dt = \int_0^\infty e^{-rt} e^{-\lambda_x^O t} \, dt = \frac{1}{r + \lambda_x^O}
\]

- **Control Variables and State Evolution:**

\( G (w) \) is the set of admissible strategies \((c, \pi, \tau)\) in the COSOCP and wealth evolution is given by

\[
dW (t) = W (t) [r + \pi (t) (\mu - r)] \, dt + W (t) \pi (t) \sigma dB (t) - c (t) \, dt
\]
Indirect Utility: General Version

Total Expected Discounted Utility \( J_{c,\pi,\tau}(w) \):

\[
E \left[ \int_0^{T_x} e^{-\delta S t} \left\{ U_1(c(t)) 1\{t\leq\tau\} + U_2 \left( \frac{W(\tau)}{\bar{a}_x+\tau} \right) 1\{t>\tau\} \right\} dt 
+ \eta e^{-\delta S T_x} \left\{ U_3 (W(T_x) + Z^s) 1\{T_x\leq\tau\} + U_3 (Z^s) 1\{T_x>\tau\} \right\} \right]
\]

with \( Z^s = Z^{prior} - \bar{Z} \)
COSOCP

General COSOCP with Exponential Mortality:

\[ V (w) = \sup_{(c,\pi,\tau) \in \mathcal{G}(w)} J_{c,\pi,\tau} (w) \quad \text{for all } w > 0 \]

\[ J_{c,\pi,\tau} (w) = E^w \left[ \int_0^\tau e^{-\beta^S t} f (c (t), W (t)) \, dt + e^{-\beta^S \tau} g (W (\tau)) \right] \]

with \( \beta^S = \delta^S + \lambda^S \) and

\[ dW (t) = W (t) [r + \pi (t) (\mu - r)] \, dt - c (t) \, dt + \sigma \pi (t) W (t) \, dB (t) \]
Verification Theorem - Optimal Strategies

- **Annuity rule**

\[
\tau^* = \inf \{ t \geq 0 | W^* (t) \notin D \}
\]

with the Continuation Region

\[
D = \{ W (t) \in G | v (W (t)) > g (W (t)) \}
\]

- **Consumption rule**

\[
c^* = I \left( v_W (W^* (t)) \right) \mathbb{1}_{\{ t \leq \tau^* \}} \text{ with } I(.) = \left( \frac{\partial f}{\partial c} (. ) \right)^{-1}
\]

- **Investment rule**

\[
\pi^* = - \frac{\mu - r}{\sigma^2} \frac{v_W (W^* (t))}{W^* (t) v_{WW} (W^* (t))} \mathbb{1}_{\{ t \leq \tau^* \}}
\]
COSOCP - Variational Inequality

The Verification Theorem Reduces the COSOCP to the Variational Inequality:

\[
\max \left\{ L_{\text{com}} v (W (t)) , g (W (t)) - v (W (t)) \right\} = 0 \quad \text{for } W (t) > 0
\]

with

\[
L_{\text{com}} v (W (t)) = \sup_{(c, \pi) \in G^\pi (W(t))} \left\{ f (c (t), W (t)) - \beta^S v (W (t)) + L v (W (t)) \right\}
\]

subject to

\[
v (W (t)) = g (W (t)) \quad \text{for all } W (t) \in \partial D
\]

and

\[
v_W (W (t)) = g_W (W (t)) \quad \text{for all } W (t) \in \partial D.
\]
Continuation Region - Properties

- Continuation Region $D$ is open and connected
- $U \subset D$ with

$$U = \{ W(t) \in \mathbb{R}^+ | L^{\text{com}}(W(t)) > 0 \}$$

- The set $U$ can be used to infer information about the form of the important continuation region $D$.
  - If $U \subsetneq D$: Continue when wealth falls out of $U$.
  - If $U = D$: Annuitise immediately when wealth exits $U$.

→ Even in the former case it is often possible to infer important information about the form of the crucial continuation region $D$ by studying the set $U$. 
Indirect Utility - Exponential Mortality

Indirect Utility Function with Exponential Mortality:

\[
E^w \left[ \int_0^\tau e^{-\beta_x t} \left\{ U_1 (c (t)) + \lambda_x^S \eta U_3 (W (t) + Z^S) \right\} \, dt \right. \\
+ e^{-\beta_x \tau} \frac{1}{\beta_x^S} \left\{ U_2 (W (\tau)) (r + \lambda_x^O) + \lambda_x^S \eta U_3 (Z^S) \right\}
\]

→ Calculate \( L^{\text{com}} g (W (t)) \) to Determine the Set \( U \)
→ Infer Information from \( U \) on the Continuation Region \( D \)
Derivation of the Set $U$ - Power Utility

Assuming Power Utility Functions We Obtain

$$L_{\text{com}}g(W(t)) = \frac{W(t)^{1-\gamma}}{1-\gamma} \left[ \gamma K_0^{-\frac{1-\gamma}{\gamma}} - \gamma K_0 K_2^{-1} \right]$$

$$+ \frac{\lambda S \eta}{1-\gamma} \left[ (W(t) + Z^s)^{1-\gamma} - (Z^s)^{1-\gamma} \right]$$

with the Constants

$$K_0 = \frac{(r + \lambda^O)^{1-\gamma}}{\beta S} > 0,$$

$$K_2 = \frac{\gamma}{\beta S - (1-\gamma) \left( r + \frac{\kappa}{\gamma} \right)}$$

$$\kappa = \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 > 0.$$
No-Bequest Case: General Results

- $U = D = \emptyset$ or $U = D = \mathbb{R}^+$ Dependent on $M^{nb}$
  - Now-or-Never Annuityisation
  - Trivial or Pure Optimal Control Problem
- $M^{nb}$ Depends on Seven Parameters in a Non-Linear Way
  \[
  M^{nb} = \gamma \left( \lambda^S + \delta^S \right)^{\frac{1-\gamma}{\gamma}} \left( r + \lambda^O \right)^{-\frac{2(1-\gamma)^2}{\gamma}} - \left( r + \lambda^O \right)^{1-\gamma} \\
  + \left( r + \frac{1}{\gamma} \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 \right) \frac{\left( r + \lambda^O \right)^{1-\gamma}}{\lambda^S + \delta^S} (1 - \gamma).
  \]
- Natural Parameter Effects
  - Risk Aversion: A+
  - Subjective Life Expectancy: A+
  - Objective Life Expectancy: A–
  - Identical Life Expectancy: A–
  - Sharpe Ratio: A– (Only Unambiguous Effect)
No-Bequest Case: Numerical Example

Minimum Sharpe Ratio for Staying in the Financial Market for Identical Life Expectancy and Risk Aversion Combinations with $\delta^S = r = 0.035$

<table>
<thead>
<tr>
<th>$E[T]$</th>
<th>$\gamma = 0.7$</th>
<th>$\gamma = 0.9$</th>
<th>$\gamma = 1.2$</th>
<th>$\gamma = 1.6$</th>
<th>$\gamma = 2$</th>
<th>$\gamma = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.5290</td>
<td>0.6000</td>
<td>0.6928</td>
<td>0.8000</td>
<td>0.8944</td>
<td>1.2649</td>
</tr>
<tr>
<td>10</td>
<td>0.3746</td>
<td>0.4242</td>
<td>0.4898</td>
<td>0.5656</td>
<td>0.6324</td>
<td>0.8944</td>
</tr>
<tr>
<td>15</td>
<td>0.3055</td>
<td>0.3464</td>
<td>0.4000</td>
<td>0.4618</td>
<td>0.5163</td>
<td>0.7302</td>
</tr>
<tr>
<td>20</td>
<td>0.2645</td>
<td>0.3000</td>
<td>0.3464</td>
<td>0.4000</td>
<td>0.4472</td>
<td>0.6324</td>
</tr>
<tr>
<td>25</td>
<td>0.2366</td>
<td>0.2683</td>
<td>0.3098</td>
<td>0.3577</td>
<td>0.4000</td>
<td>0.5656</td>
</tr>
<tr>
<td>30</td>
<td><strong>0.2160</strong></td>
<td>0.2449</td>
<td>0.2828</td>
<td>0.3265</td>
<td>0.3651</td>
<td>0.5163</td>
</tr>
</tbody>
</table>

→ Risk Aversion: A+
→ Identical Life Expectancy: A−
Assuming $\mu = 0.08$ and $\sigma = 0.2$ Implies Sharpe Ratio of 0.225.
⇒ Annuity is Chosen in Most Parameter Settings
Bequest Case $\gamma < 1$, $Z^s = 0$: General Results

- **Now-or-Never Annuitisation**: $M^b = M^{nb} + \lambda^S \eta$
- **Slight Tendency for the Financial Market** → Important Inclusion of Bequest Motive (A–)
- **Natural Parameter Effects**
- **Natural Comparison to No-Bequest Case**
  - $\frac{c^b}{W^b} < \frac{c^{nb}}{W^{nb}}$
  - $W^b > W^{nb}$
  - $\pi^b = \pi^{nb}$
  - $\frac{c^b}{W^b}$ decreases in $\eta$


Bequest Case $\gamma < 1, Z^s = 0$: Numerical Ex. (1)

Minimum Sharpe Ratio for Staying in the Financial Market for Different Combinations of the Identical Life Expectancy and the Bequest Motive for $\gamma = 0.8$

<table>
<thead>
<tr>
<th>$E[T]$</th>
<th>$\eta = 0$</th>
<th>$\eta = 0.25$</th>
<th>$\eta = 0.5$</th>
<th>$\eta = 0.75$</th>
<th>$\eta = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.5656</td>
<td>0.4409</td>
<td>0.2623</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>10</td>
<td>0.4000</td>
<td>0.3459</td>
<td>0.2817</td>
<td>0.1977</td>
<td>0.0000</td>
</tr>
<tr>
<td>15</td>
<td>0.3265</td>
<td>0.2919</td>
<td>0.2526</td>
<td>0.2059</td>
<td>0.1449</td>
</tr>
<tr>
<td>20</td>
<td>0.2828</td>
<td>0.2570</td>
<td>0.2284</td>
<td>0.1955</td>
<td>0.1559</td>
</tr>
<tr>
<td>25</td>
<td>0.2529</td>
<td>0.2322</td>
<td>0.2094</td>
<td>0.1838</td>
<td>0.1539</td>
</tr>
<tr>
<td>30</td>
<td>0.2309</td>
<td>0.2134</td>
<td>0.1942</td>
<td>0.1730</td>
<td>0.1488</td>
</tr>
<tr>
<td>35</td>
<td>0.2138</td>
<td>0.1985</td>
<td>0.1819</td>
<td>0.1637</td>
<td>0.1431</td>
</tr>
</tbody>
</table>

→ Bequest Motive: A–
→ Identical Life Expectancy: Normally A–
⇒ Slight Tendency for the Financial Market
Bequest Case $\gamma < 1$, $Z^s = 0$: Numerical Ex. (2)

Mortality Rate Transformation: $\lambda^S = \lambda^O (1 + l)$

Minimum Markup Parameter $l$ for Staying in the Financial Market for Objective Life Expectancy and Bequest Motive Combinations Assuming $\gamma = 0.8$

<table>
<thead>
<tr>
<th>$E^O [T]$</th>
<th>$\eta = 0$</th>
<th>$\eta = 0.25$</th>
<th>$\eta = 0.5$</th>
<th>$\eta = 0.75$</th>
<th>$\eta = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.4089</td>
<td>0.4191</td>
<td>0.0379</td>
<td>$-0.1760$</td>
<td>$-0.3161$</td>
</tr>
<tr>
<td>10</td>
<td>1.3395</td>
<td>0.5557</td>
<td>0.1760</td>
<td>$-0.0573$</td>
<td>$-0.2179$</td>
</tr>
<tr>
<td>15</td>
<td>1.2292</td>
<td>0.5359</td>
<td>0.1606</td>
<td>$-0.0826$</td>
<td>$-0.2557$</td>
</tr>
<tr>
<td>20</td>
<td>1.0681</td>
<td>0.4167</td>
<td>0.0337</td>
<td>$-0.2308$</td>
<td>$-0.4348$</td>
</tr>
<tr>
<td>25</td>
<td>0.8316</td>
<td>0.1666</td>
<td>$-0.3529$</td>
<td>$-0.5449$</td>
<td>$-0.6189$</td>
</tr>
<tr>
<td>30</td>
<td>0.4180</td>
<td>$-0.2325$</td>
<td>$-0.3708$</td>
<td>$-0.4780$</td>
<td>$-0.5646$</td>
</tr>
<tr>
<td>35</td>
<td>0.0855</td>
<td>$-0.1321$</td>
<td>$-0.2901$</td>
<td>$-0.4126$</td>
<td>$-0.5117$</td>
</tr>
</tbody>
</table>

→ Bequest Motive: $A$–
→ Markup Parameter $l$: $A$–
⇒ Pensioner Rejects Even Some Favourable Annuities
Bequest Case $\gamma > 1$, $Z^s > 0$: General Results

- Never Annuitisation or Wealth-Dependent Annuitisation with $D = (W, \infty)$

- Natural Comparison to No-Bequest Case

- Real COSOCP with $D = (W, \infty)$:
  - More Involved Problem
    - Simplification via Duality Arguments
    - Free Boundary Value Problem
    - Numerical Solution Algorithm
      → Boundaries
      → Value Function

- Natural Parameter Effects:
  → Life Insurance: $A+$
  → Bequest Motive: $A-$

- Heavy Consumption Smoothing
- More Aggressive Investment Rule Compared to Merton
  → Additional Option of Annuitisation
Bequest Case $\gamma > 1$, $Z^S > 0$: Problem

The Problem Involves Solving $L^{\text{com}} u(W(t)) = 0$

\[ 0 = \frac{\gamma}{1 - \gamma} [u_W(W(t))] - \frac{1 - \gamma}{\gamma} + \lambda^S \eta (W(t) + Z^S)^{1 - \gamma} \]

\[ -\beta^S u(W(t)) + rW(t) u_W(W(t)) - \kappa \frac{[u_W(W(t))]^2}{u_W W(W(t))}. \]

→ Highly Non-Linear ODE for $u$

Using Duality Arguments We Obtain

\[ 0 = \frac{\gamma}{1 - \gamma} y(t) - \frac{1 - \gamma}{\gamma} + \lambda^S \eta [-\tilde{u}_y(y(t)) + Z^S]^{1 - \gamma} \]

\[ -\beta^S \tilde{u}(y(t)) + \tilde{u}_y(y(t)) y(t) (\beta^S - r) + \kappa \tilde{u}_{yy}(y(t)) y(t)^2. \]

→ Only Slightly Non-Linear ODE for $\tilde{u}$
Bequest Case $\gamma > 1, Z^S > 0$: Solution Algorithm

- No Analytical Solution to the ODE for $\tilde{u}$ on the Dual Continuation Region $\tilde{D} = (0, y)$.
  - Numerical Solution Algorithm
  - Need Explicit Boundary Value Conditions
  - Use Smooth Paste and Smooth Fit Condition

- Problem: Thresholds $y$ and $W$ Unknown (FBVP)
  - Algorithm Must Simultaneously Solve for the Dual Function $\tilde{u}$ and the Boundaries $y$ and $W$
  - Central Idea: Probability of Annuitisation (Wealth Falling Below $W$) Vanishes as $W \to \infty$
  - Constant Merton Investment Rule in the Limit
  - Exploiting this Condition We Can Construct an Iterative Algorithm on the Bisection Method
Bequest Case $\gamma > 1, Z^S > 0$: Investment

Figure 1: Investment Rule for Different Bequest Parameters Assuming a Subjective and Objective Life Expectancy of 20 Years, $\delta^S = r = 0.035, \mu = 0.08, \sigma = 0.2$, Life Insurance Net of Subsistence of 500 and $\gamma = 2$. 
Figure 2: Consumption Fraction for Different Bequest Parameters
Assuming a Subjective and Objective Life Expectancy of 20 Years,
\( \delta^S = r = 0.035, \mu = 0.08, \sigma = 0.2, \) Life Insurance Net of Subsistence of 500 and \( \gamma = 2. \)
Main Conclusions

- COSOCP: New Solution Method
- Economically Important Risk Aversion $\gamma > 1$
- Longevity Risk Is Absolutely Relevant
  → Modelling of Lifetime
  → Role of Pension Funds
- Essential Inclusion of a Bequest Motive
  → Consumption-Wealth Trade-off
  → Absurd Strong Tendency for the Annuity Market Vanishes
Thank You Very Much for Your Attention!
Duality Arguments (1)

- Definition of the Convex Dual of $u(W(t))$

$$\tilde{u}(y(t)) = \max_{W(t) > 0} [u(W(t)) - y(t)W(t)]$$

$$\to u_W(W^*(t)) = y(t)$$

$$\to W^*(t) = I(y(t)) \text{ with } I: \text{Inverse Function of } u_W(W(t))$$

- Implied Expression for the Convex Dual

$$\tilde{u}(y(t)) = u(W^*(t)) - y(t)W^*(t) = u(I(y(t))) - y(t)I(y(t))$$

- First Order Derivative

$$\tilde{u}_y(y(t)) = u_W(I(y(t)))I_y(y(t)) - I(y(t)) - y(t)I_y(y(t))$$

$$= y(t)I_y(y(t)) - I(y(t)) - y(t)I_y(y(t))$$

$$= -I(y(t)) < 0$$
Duality Arguments (2)

- It Follows that

\[ W^*(t) = -\tilde{u}_y(y(t)) \]

- Second Order Derivative

\[ \tilde{u}_{yy}(y(t)) = -I_y(y(t)) = -\frac{1}{u_{WW}(I(y(t)))} > 0 \]

\[ \rightarrow \text{Strict Convexity of the Dual Function } \tilde{u} \text{ Follows From the Strict Concavity of the Primal Function } u. \]

- Lastly, We Have

\[ u(W^*(t)) = \tilde{u}(y(t)) + y(t)W^*(t) = \tilde{u}(y(t)) - \tilde{u}_y(y(t))y(t) \]
References (1)


[14] D. Schiess, Optimal Strategies During Retirement, Center for Finance Working Paper No. 70, University of St. Gallen, Switzerland