Approximation of multi-scale elliptic problems using patches of finite elements

R. Glowinski$^{(a)}$, J. He$^{(a)}$, A. Lozinski$^{(b)}$, J. Rappaz$^{(b)}$, J. Wagner$^{(b,c)}$

(a) Dept. of Mathematics, University of Houston, 4800 Calhoun Road, Houston, Texas 77204-3008, USA
(b) Section of Mathematics, Swiss Federal Institute of Technology, 1015 Lausanne, Switzerland
(c) Lecturer and corresponding author, e-mail: joel.wagner@epfl.ch

1 Introduction

The objective of this communication is to present a new method introduced in [1, 2] for the numerical solution of elliptic problems such that a better precision on the solution is needed in certain regions of the domain wherein the equations are to be solved. The approximation of this type of problems with multi-scale data can be approached using various methods. The technique we present here uses multiple levels of not necessarily nested grids. It is a Schwarz type domain decomposition method with complete overlapping. The proposed algorithm consists in solving the problem on a domain wherein we consider patches of elements in the regions where we would like to obtain more accuracy. Thus we calculate successively corrections in the patches to the solution obtained on a coarse grid. The discretizations of the latter are not necessarily conforming. The method resembles the Fast Adaptive Composite grid method or possibly a hierarchical method with a mortar method. However it is of much more flexible use in comparison to the latter.

A motivation for developing such a method can be found, for example, in air quality management. Pollution emission sources, and in particular point source plumes, give rise to models needing careful examination of the space-scale. Getting an accurate simulation on large scales is linked to a simulation in subregions around the pollution sources using finer grids. Such a method can be applied to boundary layer problems through the use of patches in critical regions.

This communication presents several results whose proofs are to be found in [2]. The reader should refer to the latter for a detailed bibliography giving extensive references.

2 Two-scale algorithm

Let $\Omega \subset \mathbb{R}^2$ be an open polygonal domain and consider a bilinear, symmetric, continuous and coercive form $a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$. The usual $H^1(\Omega)$-norm is equivalent to the $a$-norm defined by $||v|| = a(v,v)^{\frac{1}{2}}$, $\forall v \in H^1_0(\Omega)$. If $f \in H^{-1}(\Omega)$, due to Riesz’ representation Theorem there exists a unique $u \in H^1_0(\Omega)$ such that

$$a(u,v) = \langle f | v \rangle, \quad \forall v \in H^1_0(\Omega), \quad (1)$$

where $\langle \cdot | \cdot \rangle$ denotes the duality $H^{-1}(\Omega) - H^1_0(\Omega)$. Let us point out that (1) is the weak formulation of a problem of type $\mathcal{L}(u) = f$ in $\Omega$, $u = 0$ on the boundary $\partial \Omega$ of $\Omega$, where $\mathcal{L}(\cdot)$ is a second order, linear, symmetric, strongly elliptic operator. An approximation of $u$ by the finite element method of order $r$ consists in introducing a regular triangulation $\mathcal{T}_H$ of $\Omega$, defining $V_H = \{ g :$
\( \Omega \rightarrow \mathbb{R} \) continuous such that \( g|_K \in P_r(K), \forall K \in \mathcal{T}_H \) and \( g = 0 \) on \( \partial \Omega \), where \( P_r(K) \) is the space of polynomials of degree \( \leq r \) on triangle \( K \in \mathcal{T}_H \), and calculating \( u_H \in V_H \) satisfying
\[
a(u_H, v) = (f|v), \quad \forall v \in V_H.
\] (2)

Consider now \( \Lambda \subset \Omega \) another open polygonal domain wherein we would like to obtain a better precision on the solution \( u \) than the one given by \( u_H \). Take note that \( \Lambda \) is not necessarily the union of several triangles \( K \) of \( \mathcal{T}_H \). Besides \( \Lambda \) can be determined in practice by an a priori knowledge or an a posteriori error estimator, for example. Let \( \mathcal{T}_h \) be a regular triangulation of \( \Lambda \) and consider \( V_h = \{ g : \Omega \rightarrow \mathbb{R} \) continuous such that \( g|_K \in P_s(K), \forall K \in \mathcal{T}_h \) and \( g = 0 \) on \( \Omega \setminus \Lambda \} \). We call \( H = \max_{K \in \mathcal{T}_H} \text{diam}(K) \) and \( h = \max_{K \in \mathcal{T}_h} \text{diam}(K) \). Setting \( V_{Hh} = V_H + V_h \) we search as approximation for \( u \) the function \( u_{Hh} \in V_{Hh} \) satisfying
\[
a(u_{Hh}, v) = (f|v), \quad \forall v \in V_{Hh}.
\] (3)

Remark that \( V_{Hh} \) is a finite dimensional subspace of \( H^1_0(\Omega) \), but observe that in practice, it is not possible to determine a finite element basis of \( V_{Hh} \) since, in principle, \( V_H \cap V_h \) is not necessarily reduced to zero. Before to show how to compute \( u_{Hh} \), we establish the following \textit{a priori} estimate:

**Proposition 1 (A priori error estimate)**

Let \( q = \max(r, s)+1 \) and suppose that the solution \( u \) of (1) is in \( H^q(\Omega) \). Then the approximation \( u_{Hh} \) to \( u \) satisfies the \textit{a priori} error estimate
\[
\|u - u_{Hh}\| \leq C \left( H^r\|u\|_{H^q(\Omega \setminus \Lambda)} + h^q\|u\|_{H^q(\Lambda)} \right),
\]
where \( C \) is a constant independent of \( H \) and \( h \).

As we have mentioned above, \textit{a priori} \( V_H \cap V_h \neq \{0\} \) and it is impossible, practically speaking, to exhibit a finite element basis of the space \( V_{Hh} \) and consequently to directly compute \( u_{Hh} \). It is the reason for which we suggest the following algorithm to compute \( u_{Hh} \).

**Algorithm 1**

1. Set \( u^0 = u_H \in V_H \) and choose \( \omega \in (0; 2) \).
2. For \( n = 1, 2, 3, \ldots \) find
   \[
   (i) \quad w_h \in V_h \text{ such that } a(w_h, v) = (f|v) - a(u^{n-1}, v), \quad \forall v \in V_h;
   u_h^{n-\frac{1}{2}} = u^{n-1} + \omega w_h;
   
   (ii) \quad w_H \in V_H \text{ such that } a(w_H, v) = (f|v) - a(u^{n-\frac{1}{2}}, v), \quad \forall v \in V_H;
   u^n = u^{n-\frac{1}{2}} + \omega w_H.
   \]

When implementing the algorithm, the coarse and the fine parts of \( u^n \) and \( u^{n-\frac{1}{2}} \) are stored separately. In practice this is efficient for calculating the scalar product \( a(\cdot, \cdot) \) in the right hand side of (i) and (ii).

3 Convergence analysis

We shall now analyze the convergence of the two-scale algorithm. If \( P_h : V_{Hh} \rightarrow V_h \) and \( P_H : V_{Hh} \rightarrow V_H \) are orthogonal projectors from \( V_{Hh} \) upon \( V_h \) and \( V_H \) respectively with regard
to the scalar product \(a(\cdot, \cdot)\), we have \(u_{Hh} - u^n = (I - \omega P_H)(I - \omega P_h)(u_{Hh} - u^{n-1})\), where \(I\) denotes the identity operator in \(V_{Hh}\). Setting

\[
B = (I - \omega P_H)(I - \omega P_h),
\]

we obtain that \(u_{Hh} - u^n = B^n(u_{Hh} - u^0)\).

We set \(V_{Hh0} = V_H \cap V_h\) and \(V_{Hh0}^\perp\) the orthogonal complement of \(V_{Hh0}\) in \(V_{Hh}\).

For \(\omega \in (0; 2)\) and \(\tilde{\gamma} \in [0; 1]\) defined by

\[
\tilde{\gamma} = \begin{cases} 
\sup_{v_h \in V_h \cap V_{Hh0}: v_h \neq 0} \frac{a(u, v_H)}{||v_H||^2 ||v_h||}, & \text{if } V_h \neq V_{Hh0} \text{ and } V_H \neq V_{Hh0}, \\
0, & \text{otherwise},
\end{cases}
\]

we introduce the functions

\[
\rho(\tilde{\gamma}, \omega) = \begin{cases} 
\frac{\omega^2 \tilde{\gamma}^2}{2} - \omega + 1 + \frac{\omega \tilde{\gamma}}{2} \sqrt{\omega^2 \tilde{\gamma}^2 - 4 \omega + 4}, & \text{if } \omega \leq \omega_0(\tilde{\gamma}), \\
0, & \text{otherwise},
\end{cases}
\]

where

\[
\omega_0(\tilde{\gamma}) = \begin{cases} 
\frac{2 - 2 \sqrt{1 - \frac{1}{\tilde{\gamma}^2}}}{\tilde{\gamma}^2}, & \text{for } \tilde{\gamma} \in (0; 1), \\
1, & \text{for } \tilde{\gamma} = 0
\end{cases}
\]

and

\[
N(\tilde{\gamma}, \omega) = \frac{1}{2} \omega (2 - \omega) \tilde{\gamma} + \sqrt{\frac{1}{4} \omega^2 (2 - \omega)^2 \tilde{\gamma}^2 + (\omega - 1)^2}.
\]

An abstract analysis of the spectral properties of the iteration operator \(B\) leads to the following result:

**Proposition 2**

1. If \(\omega \in (0; 2)\), then the algorithm (i), (ii) converges, i.e. \(\lim_{n \to \infty} ||u^n - u_{Hh}|| = 0\).
2. The spectral norm of \(B\) induced by the scalar product \(a(\cdot, \cdot)\) is given by \(\|B\| = N(\tilde{\gamma}, \omega) < 1\), when \(\omega \in (0; 2)\).
3. The spectral radius of \(B\) is given by \(\rho(B) = \rho(\tilde{\gamma}, \omega) < 1\), when \(\omega \in (0; 2)\).

Thus we have the convergence of Algorithm 1 when \(\omega \in (0; 2)\) and an upper bound for the convergence factor in the norm \(|| \cdot ||\).

The third result of Proposition 2 gives an algebraic relationship of the spectral radius \(\rho\) of the operator \(B\) as function of \(\tilde{\gamma}\) and \(\omega\). This leads to a very convenient method to determine numerically a good approximation for \(\tilde{\gamma}\). Running Algorithm 1 for given \(\omega\) we can evaluate an estimate of \(\rho\) by taking the relationship \(\rho(\tilde{\gamma}, \omega) = \lim_{n \to \infty} \sqrt[3]{||u^n||}\), and hence find an estimate of the parameter \(\tilde{\gamma}\). Note that in the case \(V_{Hh0} = \{0\}\), \(\tilde{\gamma}\) corresponds to the constant of the strengthened Cauchy-Buniakowski-Schwarz inequality.

### 4 Numerical results

The algorithm is illustrated on a Poisson-Dirichlet problem in the domain \(\Omega = (-1;1)^2\) with the right-hand side \(f = -4\eta \chi(R) \frac{R^2 + R^4 - x^2}{|x^2 - R^2|^4} \exp \left(\frac{1}{x^2} \exp \left(\frac{-1}{|x^2 - R^2|}\right)\right)\), where \(R = \sqrt{x_1^2 + x_2^2}\) and
\( \chi(R) = 1 \) if \( R \leq \epsilon \), \( \chi(R) = 0 \) if \( R > \epsilon \); we take \( \eta = 10 \) and \( \epsilon = 0.5 \). Away from the origin \((0, 0)\) the solution is smooth. In a region close to \((0, 0)\) where the solution is peaking, we apply a patch with a finer mesh. For the triangulation of \( \Omega \), we use a coarse uniform grid with mesh size \( H \) and \( r = 1 \). We consider a patch \( \Lambda \) with a fine uniform triangulation of size \( h \) and \( s = 1 \). We consider two cases of patches and grids. In a first constellation we take both grids structured. We choose \( \Lambda = \left(-\frac{1}{4} + kh; \frac{1}{4} + kh\right]^2 \) and the mesh sizes \( H \) and \( h \) such that the fine triangulation is nested in the coarse one for \( k = 0 \). Varying the parameter \( k \) induces a translation of the patch \( \Lambda \) hence leading to situations with non-nested grids. A second constellation of interest is where both grids are unstructured and \( \Lambda = \left(-\frac{1}{4}, \frac{1}{4}\right)^2 \).

An illustration of the convergence of the algorithm (\( \omega = 1 \)) for some used grid types is given on Figure 1(a). Figure 1(b) shows the convergence in the mesh size.

More detailed results can be found in [2] including extensive illustrations of the numerical evaluation of \( \hat{\gamma} \). Constant \( \hat{\gamma} \) is the key ingredient in the investigation of the convergence speed of the algorithm. In the cases illustrated on Figure 1(a) we have the following approximations: constant \( \hat{\gamma} = 0.306 \) in the nested case with \( k = 0 \), \( \hat{\gamma} = 0.785 \) with \( k = 2 \), \( \hat{\gamma} = 0.474 \) with \( k = 4 \), and \( \hat{\gamma} = 0.944 \) in the case of unstructured grids.

The algorithm is easily generalized and proved to converge in the case of more than two levels (see [2]). At each level of refinement the correction in all non-overlapping patches can be parallelized. Using patches to connect problems in adjacent domains with non-conforming grids leads to efficient coupling techniques. Large domains can be split into more non-overlapping subdomains for parallel treatment and patches are to be used at a second level to connect the subdomains.

References
