Linear Duality, Term Structure, and Valuation

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1 Introduction

The paper's objective is to interpret no-arbitrage conditions by means of linear programming. Basic statements about the term structure of a market with frictions can be derived using the relation of primal and associated dual programs. The duality concept applies mutatis mutandis to the valuation of cash flows from an individual investor's point of view.

The results themselves presented in this paper have been seen before; they can be found in DERMONDY and ROCKAFELLAR [2], for instance. Recently, the ideas have been applied to the Swiss federal bond market in an empirical study by TOBLER [6] aiming at identifying the market's term structure. What is new here is the rigorous approach by linear programming and its duality theory. To each of the three no-arbitrage conditions a linear program is assigned whose dual program immediately gives rise to the desired existence statements; concise proofs result.

Before getting started, the fundamental duality theorem is recalled.

Theorem 1. *Let n, m be integers, let $A \in \mathbb{R}^{n \times m}$ be a matrix, and let $b, u \in \mathbb{R}^n$ and $c, v \in \mathbb{R}^m$ be vectors. The linear program $\max b^t u$ subject to $A^t u \leq c$ and $u \geq 0$ has an optimal solution $\hat{u}$ if and only if its dual program $\min c^t v$ subject to $Av \leq b$ and $v \geq 0$ has an optimal solution $\check{v}$, in which case the extremal values $b^t \hat{u} = c^t \check{v}$ coincide.* $\square$

Note that a linear program is said to be solvable (or, equivalently, to have an optimal, finite or proper solution) if the constraint region is non-empty and if the objective function attains a finite extremal value. A proof of the classic result is contained in any standard textbook on linear programming; a recent example is the one PADBERG [4], a forthcoming the one of DANTZIG and THAPA [1].

2 Term structure

Consider a finite number of discrete cash flows. If n denotes the number of cash flows and m the number of future payment dates, then the investment opportunities are given by a matrix $C = (C_0 \quad \tilde{C}) \in \mathbb{R}^{n \times (1+m)}$, where $C_0 \in \mathbb{R}^n$ is today's price vector and $\tilde{C} \in \mathbb{R}^{n \times m}$ the payoff matrix.

As far as individual investment strategies are concerned, attention is restricted to buy and hold strategies (e.g. bonds are held until maturity). Therefore, investments are characterized by their initial portfolio $x' \in \mathbb{R}^n$; positive and negative entries represent long and short positions, respectively.

What one is interested in is to capture the current price structure, to realize whether arbitrage is possible or not, and to price arbitrary cash flows. Arbitrage refers to an investment strategy with positive (or non-negative) today's income and a non-negative future payoff schedule.

Economically, prices should reflect the net present values of the underlying cash flows. Thus, price or *term structure* refers to a vector $\tilde{\delta} \in \mathbb{R}^m$ of discounting factors satisfying, ideally, $\tilde{C}\tilde{\delta} = C_0$. No-arbitrage assumptions permit to deduce its existence along with several properties.

Normally, future positive income has a positive or at least non-negative present value. And the earlier an income flows, the higher its net present value should be. Therefore, it seems reasonable to expect positive or at least non-negative factors decreasing with time. To this end, besides the portfolio positions $x' \in \mathbb{R}^n$ referring to the cash flows C , one explicitly considers additional positions $z \in \mathbb{R}^m$ with $z \geq 0$ referring to holdovers

$$Z = (Z_0 \quad \tilde{Z}) = \begin{pmatrix} 1 & 1 & & & 0 \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 1 \end{pmatrix} \in \mathbb{R}^{m \times (1+m)}.$$

Arguing gains momentum if the common hypothesis of absence of frictions characteristic for a perfect market is relaxed. There might be transaction costs and taxes to consider as well as bid-ask spreads, for instance. Assuming proportionality of the necessary adjustments to the volume, the cash flow matrix $C = (C_0 \quad \tilde{C})$ gives rise to matrices $X = (X_0 \quad \tilde{X})$ and $Y = (Y_0 \quad \tilde{Y})$ in $\mathbb{R}^{n \times (1+m)}$ applicable with long and short positions $x, y \geq 0$ (previously collected as $x' = x \ y$).

Note that the described multi-period setting with one single outcome state per period may be reinterpreted as a one-period investment problem with n assets and m different outcome states. Assigning probabilities to the outcome states may overcome the deterministic setting. The theory stays very much the same.

3 No-arbitrage conditions

The *weak* no-arbitrage condition excludes the opportunity of arbitrage profit just in time with the portfolio decision. In addition, the *strong* no-arbitrage condition excludes arbitrage profit at any future point of time. And even beyond, the *complete* no-arbitrage condition excludes any further portfolio transaction resulting in non-negative payoff. The formal definitions with regard to the introduced notation read as follows.

Definition 1. The weak no-arbitrage condition (WNA) is satisfied if there exists no portfolio $(x, y, z) \neq 0$ with

$$\begin{aligned} X_0^t x + Y_0^t y - Z_0^t z &> 0, \\ \tilde{X}^t x - \tilde{Y}^t y + \tilde{Z}^t z &= 0. \end{aligned}$$

Definition 2. The strong no-arbitrage condition (SNA) is satisfied if there exists no portfolio $(x, y, z) \neq 0$ with

$$X^t x - Y^t y + Z^t z = 0$$

apart from one with

$$X^t x - Y^t y + Z^t z = 0.$$

Definition 3. The complete no-arbitrage condition (CNA) is satisfied if there exists no portfolio $(x, y, z) \neq 0$ with

$$X^t x - Y^t y + Z^t z = 0$$

apart from $(x, y, z) = 0$.

It is mentioned that there is an alternative notion of arbitrage opportunities of *first* and *second type* used by INGERSOLL [3], for instance. Omitting a definition here, its characterization in terms of weak and strong no-arbitrage is briefly mentioned: while the weak no-arbitrage condition is equivalent to the absence of arbitrage opportunities of the second type, the strong no-arbitrage condition holds if and only if there exists neither an arbitrage opportunity of the first type nor one of the second type. Much more insight than such tautologous reasoning offers the dual point of view, however.

Proposition 1. The following statements are equivalent:

- (1) The weak no-arbitrage condition holds.
- (2) The linear program (WNA) is solvable.

$$\begin{aligned} \max \quad & X_0^t x + Y_0^t y - Z_0^t z \\ \text{s.t.} \quad & \tilde{X}^t x - \tilde{Y}^t y + \tilde{Z}^t z = 0, \\ & x, y, z \geq 0 \end{aligned} \tag{WNA}$$

- (3) There exists a $\delta \in \mathbb{R}^{1+m}$ with $1 = \delta_0 - \delta_1 - \dots - \delta_m \geq 0$ s.t. $X\delta \leq 0 \leq Y\delta$.

Proof. While the equivalence of the first two statements follows immediately from the definition of weak no-arbitrage, the linear program is solvable (i.e. it has an optimal solution with finite objective value) if and only if the associated dual program

$$\begin{aligned} \min \quad & 0^t \tilde{\delta} \\ \text{s.t.} \quad & \tilde{X} \tilde{\delta} \leq X_0, \\ & \tilde{Y} \tilde{\delta} \geq Y_0, \\ & \tilde{Z} \tilde{\delta} \leq Z_0, \\ & \tilde{\delta} \geq 0 \end{aligned} \tag{WNA}$$

is solvable, which translates into the third statement via $\delta = (1 \ \tilde{\delta})$. $\square$

Let Δ denote the compact polyhedron of vectors $\delta \in \mathbb{R}^{1+m}$ with the properties referred to in the above third statement.

Proposition 2. *The following statements are equivalent:*

- (1) *The strong no-arbitrage condition holds.*
- (2) *The linear program (SNA) is solvable.*

$$\begin{aligned} \max \quad & \mathbf{1}^t(X^t x - Y^t y) \\ \text{s.t.} \quad & X^t x - Y^t y + Z^t z = 0, \\ & x, y, z \geq 0 \end{aligned} \tag{SNA}$$

- (3) *There exists a $\delta \in \mathbb{R}^{1+m}$ with $1 = \delta_0, \delta_1, \dots, \delta_m > 0$ s.t. $X\delta \leq 0 \leq Y\delta$.*

Proof. The statements hold if and only if the associated dual program

$$\begin{aligned} \min \quad & 0^t \delta \\ \text{s.t.} \quad & X\delta \leq X\mathbf{1}, \\ & Y\delta \geq Y\mathbf{1}, \\ & Z\delta \leq Z\mathbf{1}, \\ & \delta \geq 0 \end{aligned} \tag{SNA}$$

is solvable. $\square$

Note that $Z\mathbf{1} = 0$. By abuse of notation, in the sequel the very same symbol $\mathbf{1}$ denotes a vector all of whose entries are one, disregarding whether it lies in $\mathbb{R}^{1+m}, \mathbb{R}^n$ or $\mathbb{R}^m$.

Proposition 3. *The following statements are equivalent:*

- (1) *The complete no-arbitrage condition holds.*
- (2) *The linear program (CNA) is solvable.*

$$\begin{aligned} \max \quad & \mathbf{1}^t(X^t x - Y^t y) + \mathbf{1}^t(x + y) + \mathbf{1}^t z \\ \text{s.t.} \quad & X^t x - Y^t y + Z^t z = 0, \\ & x, y, z \geq 0 \end{aligned} \tag{CNA}$$

- (3) *There exists a $\delta \in \mathbb{R}^{1+m}$ with $1 = \delta_0 > \delta_1 > \dots > \delta_m > 0$ s.t. $X\delta < 0 < Y\delta$.*

Proof. The statements hold if and only if the associated dual program

$$\begin{aligned} \min \quad & 0^t \delta \\ \text{s.t.} \quad & X\delta \leq X\mathbf{1} - \mathbf{1}, \\ & Y\delta \geq Y\mathbf{1} - \mathbf{1}, \\ & Z\delta \leq Z\mathbf{1} - \mathbf{1}, \\ & \delta \geq 0 \end{aligned} \tag{CNA}$$

is solvable. $\square$

Note that the set of vectors $\delta \in \mathbb{R}^{m+1}$ with the above properties derived from the complete no-arbitrage condition is precisely the interior of the closed set Δ defined with regard to weak no-arbitrage.

4 Induced valuation

No-arbitrage conditions are based on the non-existence of portfolios replicating at least zero cash flows. Portfolios replicating at least a given non-zero cash flow may be used to value this cash flow.

Definition 4. *The induced long value of a cash flow $c \in \mathbb{R}^m$ is defined as*

$$\begin{aligned} L(c) = \max \quad & X_0^t x + Y_0^t y - Z_0^t z \\ \text{s.t.} \quad & \tilde{X}^t x - \tilde{Y}^t y + \tilde{Z}^t z \leq c, \\ & x, y, z \geq 0, \end{aligned}$$

the induced short value of a cash flow $c \in \mathbb{R}^m$ is defined as

$$\begin{aligned} S(c) = \max \quad & X_0^t x + Y_0^t y - Z_0^t z \\ \text{s.t.} \quad & \tilde{X}^t x - \tilde{Y}^t y + \tilde{Z}^t z \leq c, \\ & x, y, z \geq 0. \end{aligned}$$

Note that the definitions give rise to functions $L, S : \mathbb{R}^m \rightarrow \mathbb{R}$ if and only if the weak no-arbitrage condition holds. Otherwise, by convention, the values are considered to be $+\infty$ and $-\infty$, respectively.

Proposition 4. *Assume that the weak no-arbitrage condition holds. Then, the induced values satisfy the following properties:*

- (1) $L(c) = \max_{\delta \in \Delta} \delta^t c$ and $S(c) = \min_{\delta \in \Delta} \delta^t c$ for all $c \in \mathbb{R}^m$,
 L and S are piecewise linear,
- (2) Δ is completely determined by L and S , respectively, i.e.

$$\begin{aligned} \Delta &= \{ \delta \in \mathbb{R}^m : \delta^t c \leq L(c) \text{ for all } c \in \mathbb{R}^m \\ &= \{ \delta \in \mathbb{R}^m : \delta^t c \geq S(c) \text{ for all } c \in \mathbb{R}^m. \end{aligned}$$

Proof. The first statement follows by linear duality. The piecewise linearity is due to the property of Δ being a compact polyhedron with a finite number of vertices. The second result exploits that the set Δ (being the intersection of some closed half-spaces) is closed convex which implies that it is the intersection of all the closed half-spaces containing it (confer e.g. ROCKAFELLAR [5]). $\square$

The characterization obtained by linear duality has immediate consequences.

Corollary 1. *Assume that the weak no-arbitrage condition holds. Then, the induced values satisfy the following properties:*

- (1) $S(c) = L(c)$, $L(0) = S(0) = 0$, $L(\lambda c) = \lambda L(c)$ and $S(\lambda c) = \lambda S(c)$,
 - (2) $S(c) \leq L(c)$, L is convex and S is concave.
 - (3) $L(c + c') \leq L(c) + L(c')$ and $S(c) + S(c') \leq S(c + c')$
- for all $c, c' \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$ with $\lambda \geq 0$. □

The case of complete no-arbitrage allows additional conclusions.

Proposition 5. *Assume that the complete no-arbitrage condition holds. Then, the induced values satisfy the following properties:*

- (1) $S(c) < L(c)$ for all $0 \neq c \in \mathbb{R}^m$,
- (2) L and S are nonlinear functions.

Proof. The inequality results applying the dual characterization of the induced values to a Δ whose interior is non-empty by complete no-arbitrage assumption. The nonlinearity, due to $L(c) + L(c) = L(c)$ $S(c) > 0$ an immediate consequence, holds even restricted to a small neighbourhood of $0 \in \mathbb{R}^m$. □

The complete no-arbitrage excludes the possibility that the set Δ is contained in a hyperplane of $\mathbb{R}^{m+1}$. Economically, this implies that under complete no-arbitrage the introduced piecewise linear valuation operators cannot be linear operators.

References

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