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Optimal Design of the Attribution of Pension Fund Performance to Employees

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Optimal Design of the Attribution of Pension Fund Performance to Employees

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Abstract

The paper analyses a defined-contribution pension fund in continuous time. According to a prespecified attribution scheme the interest rate paid on the employees' accounts is a linear function of the fund's investment performance. An attribution scheme consists of a participation rate and an intercept. For each attribution scheme the pension fund maximises the expected utility of the funding ratio at the end of a planning horizon and the employees derive utility from their savings accounts at the time they exit the plan. Solving the optimisation problem of the pension fund leads to constant optimal investment strategies. For the pension fund and the employees, respectively, indirect utility functions can be derived on the set of attributions. It turns out that all Pareto-optimal attribution schemes are characterised by the same optimal participation rate. As a main result, we derive the total welfare gain - measured by the increase in appropriate certainty equivalents of the pension fund and the employees, respectively - that installs from replacing no participation with optimal participation. For reasonable parameter values a substantial increase in the risk-adjusted rate of return on employees' accounts can be achieved if the welfare gain is fully attributed to the employees.

Keywords: Optimal Design of Investment Performance Participation; Optimal Investment; Pension Fund; Pareto-Optima; Welfare Gain

1 Introduction

In pension finance one has to distinguish between defined-benefit and defined-contribution plans. In a defined-benefit plan benefits are defined in advance and the plan sponsor and the employees have to adjust their contributions. In the literature optimal investment strategies for defined-benefit plans are treated by [14], [5], [7] and [16] among others. Furthermore, the articles by [24], [22], [17] and [18] deal with portfolio optimisation in the asset-liability context. Since the plan sponsor has to bear substantial financial and actuarial risk and since the valuation of accrued retirement benefits of employees changing their plan is difficult, defined-benefit plans have become less and less popular. In a defined-contribution plan in the narrow sense the accrued retirement benefits of employees are invested and the positive or negative

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fund return is fully attributed to the employees' accounts. At retirement each employee obtains the final amount from his account as a cash payment or as a corresponding pension, respectively. However, there is a wide span of different defined-contribution plans. On the one hand, there exist pure defined-contribution plans like the US 401k-plans where the employee is free to choose an individual investment policy. The pension fund consequently bears no financial risk at all and acts only as a broker between employees and financial and life insurance markets. Hence, employees may be considered as individual investors. Authors dealing with this topic in continuous time are [19], [1], [11] and [12]. In discrete time there are articles by [23] and [15] among others. On the other hand, there exist defined-contribution pension plans with certain guaranteed benefits. The plan sponsor bears substantial financial risk in order to protect the employees against fluctuations of financial markets (e.g. the mandatory part of Swiss pension funds displays guaranteed benefits like a guaranteed attribution rate). In the literature [4], [8] and [9] analyse a defined-contribution plan providing a guarantee. The present paper concentrates on the optimal design of the attribution of the pension fund performance to its employees.

If a pension plan increases the funding ratio (ratio of the value of assets to the present value of net obligations) during prospering financial markets and decreases it in declining financial markets then a risk transfer between different generations of employees can be established. This idea was formalised by [3]. In their model they introduced an intertemporal risk transfer by assuming that the rate attributed to the employees' accounts depends on the funding ratio but not on the fund's investment performance. They showed that in such a framework the pension fund can choose investment policies such that all employees would be worse off if they acted as individual investors. Furthermore, there are articles deriving optimal investment strategies for defined-contribution plans which put the funding ratio into the focus of the optimisation like [6], [10] and [20]. In contrast to these articles, the liabilities in the papers by [21] and [3] are not exogeneous. [3] models the return attributed to the employees' accrued retirement benefits as a function of the funding ratio. In this paper we analyse the dependency on the fund's investment performance. A similar idea was already pursued in the work of [2]. However, the present paper shows that restricting the attribution to linear functions of the fund's investment performance leads to closed-form solutions for the Pareto-optimal investment strategy, the optimal participation rate and the welfare gain resulting from optimal participation.

In this paper we deal with financial risk sharing between the defined-contribution plan sponsor and the employees in continuous time. We assume that the rate attributed to employees' accounts is a linear function of the fund's investment performance. Such an attribution scheme can be characterised by the intercept and the participation rate. Given an attribution scheme the pension fund maximises its expected utility by choosing a corresponding investment policy. Hence, for each attribution scheme a corresponding expected utility is obtained. This defines the indirect utility function of the pension fund on the set of attribution schemes. Given the investment policies of the pension fund one may calculate the expected utility of the employees for each attribution scheme. In this way the indirect utility function of the employees can be derived as well. The set of Pareto-optimal attribution schemes will be calculated and it will be shown that all Pareto-optima are characterised by the same participation rate. A formula for this optimal participation rate will be derived. Afterwards, we will define the welfare of the pension fund and the employees, respectively, by making use of appropriate certainty equivalents. Finally, a comparison of optimal participation to no participation shows that there is a substantial welfare gain for reasonable parameter values.

The paper is organised as follows. Section 2 presents the model setup by defining the financial market as well as the objective function of the pension fund and the employees, respectively. Moreover, linear attribution schemes are introduced and the evolution of assets and liabilities are developed and taken together yielding the dynamics of the funding ratio. In Section 3 we maximise the expected utility of the pension fund for a given attribution scheme which leads to a constant optimal investment strategy. We discuss the optimal investment rule and close the section with the derivation of the optimally controlled funding ratio dynamics. In Section 4 we derive the indifference curves of the pension fund and the employees, respectively, which leads to a discussion of risk-adjusted rates of return. Section 5 then presents the result that there is a unique Pareto-optimal participation rate $\alpha^* \in (0,1)$. Exploiting this participation rate the Pareto-optimal investment strategy will be derived. Furthermore, we close the section by discussing the parameter sensitivities of optimal participation and investment, respectively. Section 6 first defines the welfare of the pension fund and the employees, respectively, by making use of appropriate certainty equivalents. Afterwards, we derive the welfare gain that installs if one replaces no participation with optimal participation. It will be shown that this welfare gain is substantial for reasonable parameter values. Finally, Section 7 summarises and gives the main conclusions.

2 Model

In the sense of an overlapping generation model at each point in time $t \in \mathbb{R}^+$ employees enter the plan. In order to simplify the analysis it is assumed hat employees entering at t immediately invest a fixed amount $X_{t,0}$. On their investment the pension fund attributes an interest rate and after a time span τ at time $t+\tau$ employees leave the plan with their accrued retirement benefits $X_{t,\tau}$.

2.1 Pension Fund

The pension fund has a riskless investment opportunity $i = 0$ and risky opportunities $i = 1, ..., N$ whose price processes are given by geometric Brownian motions, i.e.

$$\frac{dS_{0,t}}{S_{0,t}} = rdt$$

and

$$\frac{dS_{i,t}}{S_{i,t}} = (r + \pi_i)\, dt + \sum_{j=1}^{N} \sigma_{ij} \mathbf{dZ}_{j,t} \qquad , i = 1, ..., N$$

where r denotes the riskless interest rate, $\pi \in \mathbb{R}^N$ the expected excess returns, $\mathbf{Z}_t$ a N-dimensional standard Brownian motion and σ is a regular matrix determining how the risky assets are driven by the Brownian motions.

At time t the financial situation of the pension fund is given by

A_t	value of assets at time t
L_t	value of liabilities at time t
$F_t = \frac{A_t}{L_t}$	funding ratio at time t.

The pension fund chooses a portfolio $\mathbf{x}_t \in \mathbb{R}^N$ in order to invest A_t. Hence, the dynamics of the value of assets are given by[1]

$$\frac{dA_t}{A_t} = \left(r + \mathbf{x}_t^T \pi\right) dt + \mathbf{x}_t^T \sigma \mathbf{dZ}_t. \tag{1}$$

On the liability side the fund attributes a return

$$dR_t = \left(r + a + \alpha \mathbf{x}_t^T \pi\right) dt + \alpha \mathbf{x}_t^T \sigma \mathbf{dZ}_t$$

[1]In principle net contributions C_t to the fund should be considered as well. This would lead to

$$\frac{dA_t}{A_t} = \left(r + \mathbf{x}_t^T \pi\right) dt + C_t dt + \mathbf{x}_t^T \sigma \mathbf{dZ}_t.$$

However, we simplify the analysis by assuming $C_t = 0$.

with $0 \leq \alpha < 1$ to the accrued retirement benefits of the employees. The *linear attribution scheme* $(\alpha, a) \in D$ is characterised by the participation rate α and the premium a.

Hence, the dynamics of the liabilities are given by

$$\frac{dL_t}{L_t} = dR_t$$

or

$$\frac{dL_t}{L_t} = \left(r + a + \alpha \mathbf{x}_t^T \pi\right) dt + \alpha \mathbf{x}_t^T \sigma \mathbf{dZ}_t. \tag{2}$$

Applying Itô's lemma to the funding ratio leads to

$$\frac{dF_t}{F_t} = \frac{dA_t}{A_t} - \frac{dL_t}{L_t} + \left(\frac{dL_t}{L_t}\right)^2 - \frac{dA_t}{A_t}\frac{dL_t}{L_t}. \tag{3}$$

Plugging (1) and (2) into (3) we get

$$\frac{dF_t}{F_t} = \left[(1-\alpha) \mathbf{x}_t^T \pi - a + \left(\alpha^2 - \alpha\right) \mathbf{x}_t^T \mathbf{V} \mathbf{x}_t\right] dt + (1-\alpha) \mathbf{x}_t^T \sigma \mathbf{dZ}_t \tag{4}$$

with $\mathbf{V} = \sigma\sigma^T$.

The pension fund maximises the expected utility of the funding ratio F_T at the planning horizon T. The relative risk aversion of the fund[2] is $R_p > 1$. Hence, for a given attribution scheme $(\alpha, a) \in D$ the objective function of the fund is given by

$$U_p\left(F_T\right) = \frac{E\left[F_T^{1-R_p}\right]}{1-R_p}. \tag{5}$$

2.2 Employees

On the employees' accounts $X_{t,s}$ at time $t+s$ the pension fund attributes a return

$$dR_{t+s} = \left(r + a + \alpha \mathbf{x}_{t+s}^T \pi\right) ds + \alpha \mathbf{x}_{t+s}^T \sigma \mathbf{dZ}_{t+s}.$$

Hence, the dynamics of the employees' accounts are given by

$$\frac{dX_{t,s}}{X_{t,s}} = \left(r + a + \alpha \mathbf{x}_{t+s}^T \pi\right) ds + \alpha \mathbf{x}_{t+s}^T \sigma \mathbf{dZ}_{t+s}. \tag{6}$$

All employees have the same constant relative risk aversion[3] $R_e > 1$ with respect to the accrued retirement benefits $X_{t,\tau}$ at the end of their working life. Hence, the objective function of the employees is given by

$$U_e\left(X_{t,\tau}\right) = \frac{E\left[X_{t,\tau}^{1-R_e}\right]}{1-R_e}. \tag{7}$$

3 Optimal Portfolio and Optimally Controlled Funding Ratio

The optimal investment policy of the pension fund can be derived with stochastic control.

PROPOSITION 3.1 *Maximising the utility of the pension fund defined in (5)*

$$U_p\left(F_T\right) = \frac{E\left[F_T^{1-R_p}\right]}{1-R_p}$$

[2] According to the results of [13] it is realistic to assume $R_p > 1$.
[3] According to the results of [13] it is realistic to assume $R_e > 1$.

subject to the funding ratio evolution given in (4)

$$\frac{dF_t}{F_t} = \left[(1-\alpha)\,\mathbf{x}_t^T\pi - a + \left(\alpha^2-\alpha\right)\mathbf{x}_t^T\mathbf{V}\mathbf{x}_t\right]dt + (1-\alpha)\,\mathbf{x}_t^T\sigma\mathbf{dZ}_t$$

leads to the optimal investment policy

$$\mathbf{x}_t = \mathbf{x}^* = \frac{1}{2\alpha + R_p\,(1-\alpha)}\mathbf{V}^{-1}\pi \quad , 0 \le t \le T. \tag{8}$$

Proof:
See Appendix A.1.

■

Remark: 3.2

1. The optimal investment policy does not depend on the planning horizon T and is constant over time. Hence, the funding ratio follows a geometric Brownian motion.

2. The optimal investment strategy is proportional to the Merton investment rule $x^M = \frac{1}{R_p}\mathbf{V}^{-1}\pi$. Moreover, we naturally obtain the Merton rule in the special case of $\alpha = 0$. Introducing risk sharing (assuming $\alpha > 0$), the pension fund invests more aggressively than the Merton investor if and only if

$$2\alpha + R_p\,(1-\alpha) < R_p$$
$$\Leftrightarrow$$
$$R_p > 2. \tag{9}$$

3. The optimal investment rule naturally depends on the variance-covariance matrix $\mathbf{V}$, the expected excess returns π, the relative risk aversion of the pension fund R_p and finally, on the participation rate α. The premium a of the attribution scheme however has no influence on optimal investment.

4. Finally, we give the sensitivity of the optimal investment rule with respect to the participation rate

$$\frac{\partial \mathbf{x}^*}{\partial \alpha} = \frac{R_p - 2}{\left[2\alpha + R_p\,(1-\alpha)\right]^2}\mathbf{V}^{-1}\pi. \tag{10}$$

Obviously, the pension fund invests more aggressively the higher the participation rate if $R_p > 2$.

□

Plugging the optimal investment rule (8) into the general funding ratio dynamics (4) we obtain the evolution of the optimally controlled funding ratio

$$\frac{dF_t}{F_t} = \mu_p dt + \sigma_p dZ_t^* \tag{11}$$

with

$$\begin{aligned}\mu_p &= (1-\alpha)\left[\frac{1}{2\alpha + R_p\,(1-\alpha)} - \frac{\alpha}{\left[2\alpha + R_p\,(1-\alpha)\right]^2}\right]\pi^T\mathbf{V}^{-1}\pi - a \\ &= (1-\alpha)\,\frac{\alpha + R_p\,(1-\alpha)}{\left[2\alpha + R_p\,(1-\alpha)\right]^2}\pi^T\mathbf{V}^{-1}\pi - a \end{aligned}\tag{12}$$

and

$$\sigma_p = \frac{1-\alpha}{2\alpha + R_p\,(1-\alpha)}\left(\pi^T\mathbf{V}^{-1}\pi\right)^{0.5}. \tag{13}$$

4 Preferences on the Set of Attributions

4.1 Indifference Curves of the Pension Fund

Exploiting (11) we obtain

$$d\ln F_t = \left(\mu_p - \frac{\sigma_p^2}{2}\right) dt + \sigma_p dZ_t^*.$$

Integration then yields

$$\ln F_T = \ln F_0 + \left(\mu_p - \frac{\sigma_p^2}{2}\right) T + \sigma_p Z_T^*.$$

Inserting this result into the objective function of the pension fund given in (5) leads to

$$\begin{aligned} U_p(F_T) &= (1-R_p)^{-1} E\left[F_T^{1-R_p}\right] \\ &= (1-R_p)^{-1} E\left[e^{(1-R_p)\ln F_T}\right] \\ &= (1-R_p)^{-1} \exp\left\{(1-R_p) E[\ln F_T] + 0.5(1-R_p)^2 Var[\ln F_T]\right\} \\ &= (1-R_p)^{-1} \exp\left\{(1-R_p)\left[\ln F_0 + \left(\mu_p - \frac{\sigma_p^2}{2}\right) T + 0.5(1-R_p)\sigma_p^2 T\right]\right\} \\ &= (1-R_p)^{-1} \exp\left\{(1-R_p)\ln F_0 + (1-R_p) T\left[\mu_p - \frac{R_p}{2}\sigma_p^2\right]\right\}. \end{aligned}$$

Since only μ_p and σ_p depend on the attribution scheme $(\alpha, a) \in D$, the preferences of the pension fund can be represented by the indirect utility function

$$V_p(\alpha, a) = \mu_p - \frac{R_p}{2}\sigma_p^2.$$

Plugging (12) and (13) into the above we obtain

$$\begin{aligned} V_p(\alpha, a) &= (1-\alpha)\frac{\alpha + R_p(1-\alpha)}{[2\alpha + R_p(1-\alpha)]^2}\pi^T \mathbf{V}^{-1}\pi - a - \frac{R_p}{2}\frac{(1-\alpha)^2}{[2\alpha + R_p(1-\alpha)]^2}\pi^T \mathbf{V}^{-1}\pi \\ &= (1-\alpha)\frac{\alpha + 0.5R_p(1-\alpha)}{[2\alpha + R_p(1-\alpha)]^2}\pi^T \mathbf{V}^{-1}\pi - a \\ &= \frac{1-\alpha}{2[2\alpha + R_p(1-\alpha)]}\pi^T \mathbf{V}^{-1}\pi - a. \end{aligned} \quad (14)$$

Thus, the indifference curves of the pension fund on the set of attributions D are given by

$$a = f_p(\alpha) = 0.5\frac{1-\alpha}{2\alpha + R_p(1-\alpha)}\pi^T \mathbf{V}^{-1}\pi - C_p \quad , C_p \in \mathbb{R}. \quad (15)$$

Obviously, the indifference curves of the pension fund are downward sloping[4] and result from each other by vertical parallel shifts (choice of the constant C_p). Figure 1 shows the indifference curves on the set of attribution schemes D for $R_p = 3$ and $\pi^T \mathbf{V}^{-1}\pi = 6.25\%$.[5]

4.2 Indifference Curves of the Employees

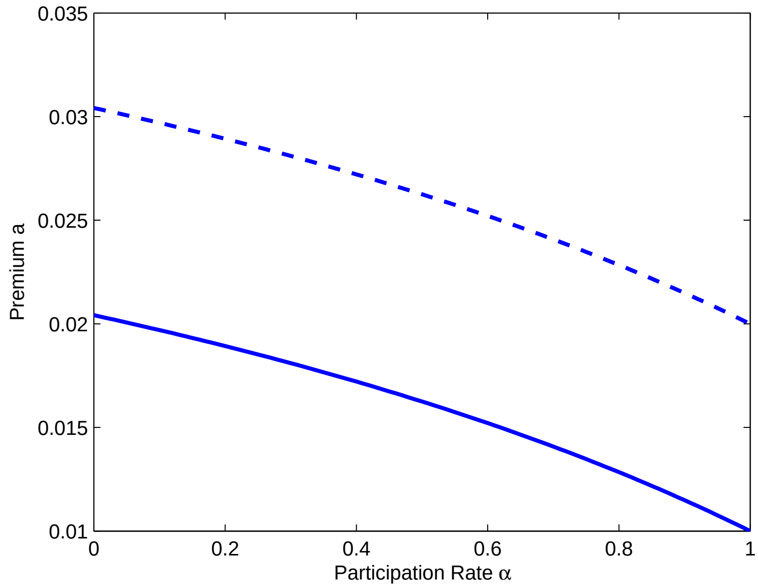
Inserting the optimal investment policy (8) into the dynamics of the employees' accounts (6) leads to

$$\frac{dX_{t,s}}{X_{t,s}} = \mu_e ds + \sigma_e dZ_{t+s}^*. \quad (16)$$

[4]See $f_p'(\alpha)$ given in (31) in Appendix A.2.

[5]Note that $\pi^T \mathbf{V}^{-1}\pi$ can be interpreted as the expected excess return of the growth optimal portfolio $\mathbf{x}^{opt} = \mathbf{V}^{-1}\pi$. Thus, assuming $\pi^T \mathbf{V}^{-1}\pi = 6.25\%$ seems quite realistic.

Figure 1: Indifference curves of the pension fund for $R_p = 3$ and $\pi^T V^{-1}\pi = 6.25\%$. The dashed curve represents $C_p = -0.02$ whereas the solid curve corresponds to $C_p = -0.01$.



with

$$\mu_e = r + a + \frac{\alpha}{2\alpha + R_p (1-\alpha)} \pi^T \mathbf{V}^{-1}\pi \tag{17}$$

and

$$\sigma_e = \frac{\alpha}{2\alpha + R_p (1-\alpha)} \left(\pi^T \mathbf{V}^{-1}\pi\right)^{0.5}. \tag{18}$$

Similarly as in Subsection 4.1 one can derive the indirect utility function of the employees as

$$\begin{aligned} V_e(\alpha, a) &= \mu_e - \frac{R_e}{2}\sigma_e^2 \\ &= r + a + \frac{\alpha}{2\alpha + R_p(1-\alpha)}\pi^T \mathbf{V}^{-1}\pi - \frac{R_e}{2}\frac{\alpha^2}{[2\alpha + R_p(1-\alpha)]^2}\pi^T \mathbf{V}^{-1}\pi \\ &= r + a + \alpha\frac{2\alpha + R_p(1-\alpha) - 0.5R_e\alpha}{[2\alpha + R_p(1-\alpha)]^2}\pi^T \mathbf{V}^{-1}\pi. \end{aligned} \tag{19}$$

Thus, the indifference curves of the employees on the set of attributions D are given by

$$a = f_e(\alpha) = -\alpha\frac{2\alpha + R_p(1-\alpha) - 0.5R_e\alpha}{[2\alpha + R_p(1-\alpha)]^2}\pi^T \mathbf{V}^{-1}\pi - r + C_e \quad , C_e \in \mathbb{R}. \tag{20}$$

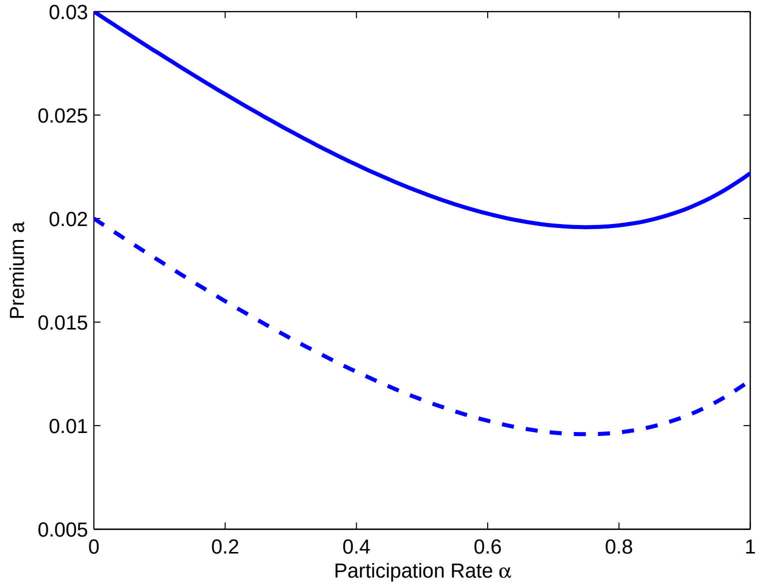
Again, the indifference curves differ from each other only by vertical shifts (choice of C_e). Figure 2 shows the indifference curves of the employees on the set of attribution schemes D for $R_p = R_e = 3$, $r = 3\%$ and $\pi^T \mathbf{V}^{-1}\pi = 6.25\%$.

5 Pareto-Efficiency and Pareto-Optimal Participation Rate

Due to the special properties (vertical parallel shifts) of the indifference curves Pareto-optimal attribution schemes $(\alpha^*, a^*) \in D$ are characterised by

$$\alpha^* = \arg\max_{\alpha\in[0,1)} [f_p(\alpha) - f_e(\alpha)].$$

Figure 2: Indifference curves of the employees for $R_p = R_e = 3$, $r = 3\%$ and $\pi^T \mathbf{V}^{-1} \pi = 6.25\%$. The dashed curve represents $C_e = 0.05$ whereas the solid curve corresponds to $C_e = 0.06$.



Remark: 5.1

1. $f_p(\alpha)$ denotes the premium the pension fund is willing to pay under the participation rate α whereas $f_e(\alpha)$ denotes the premium required by the employees for a given α.

2. The Pareto-optimal set is obtained by different choices of the premium a^*, i.e. the Pareto frontier is a vertical line in the α-a-plane.

3. α^* will be called the optimal participation rate in the remainder.

4. Evaluating $f_p'(\alpha)$ and $f_e'(\alpha)$ given in Appendix A.2 in (31) and (32), respectively, in the no participation case we obtain

$$f_p'(0) = -\frac{1}{R_p^2} \pi^T \mathbf{V}^{-1} \pi > f_e'(0) = -\frac{1}{R_p} \pi^T \mathbf{V}^{-1} \pi. \tag{21}$$

5. Figure 3 shows different indifference curves of the pension fund and the employees, respectively, on the set of attribution schemes D for $R_p = R_e = 3$, $r = 3\%$ and $\pi^T \mathbf{V}^{-1} \pi = 6.25\%$.

□

Assumption 5.2

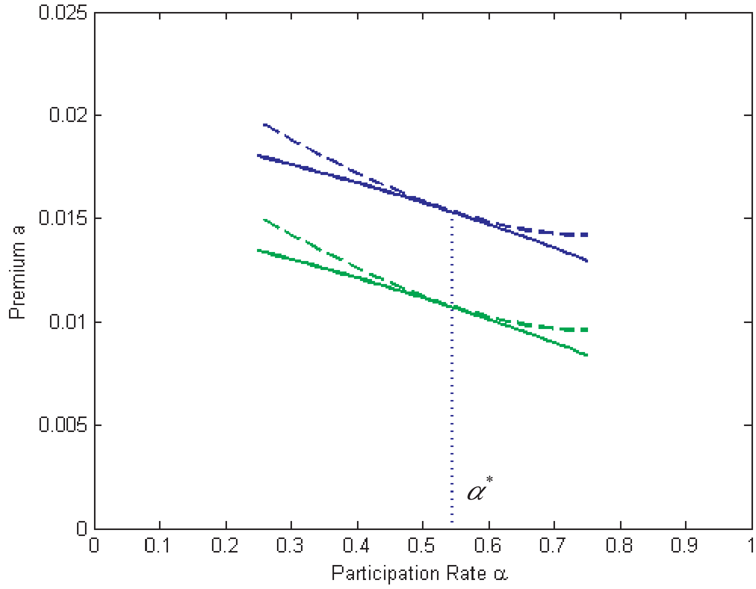
$$R_p (2 - R_e) < 2$$

Remark: 5.3

By imposing Assumption 5.2 one excludes that employees with a rather low relative risk aversion R_e belong to a pension plan with a high relative risk aversion R_p. Figure 4 shows the pairs (R_e, R_p) of the relative risk aversion of the pension fund and the employees, respectively, where Assumption 5.2 holds. Obviously, imposing Assumption 5.2 is not really restrictive.

□

Figure 3: Indifference curves of the pension fund (solid curves) and the employees (dashed curves), respectively, for $R_p = R_e = 3$, $r = 3\%$ and $\pi^T \mathbf{V}^{-1} \pi = 6.25\%$.



Proposition 5.4 *Let Assumption 5.2 be given. Then, the optimal participation rate* α^* *satisfies* $0 < \alpha^* < 1$ *and is given by*

$$\alpha^* = \frac{R_p\,(R_p - 1)}{(R_e + R_p - 3)\,R_p + 2} \tag{22}$$

and the Pareto-optimal investment strategy is given by

$$\mathbf{x}^{Pa*} = \frac{(R_e + R_p - 3)\,R_p + 2}{R_p^2 R_e} \mathbf{V}^{-1} \pi. \tag{23}$$

Proof:

See Appendix A.2.

■

Remark: 5.5

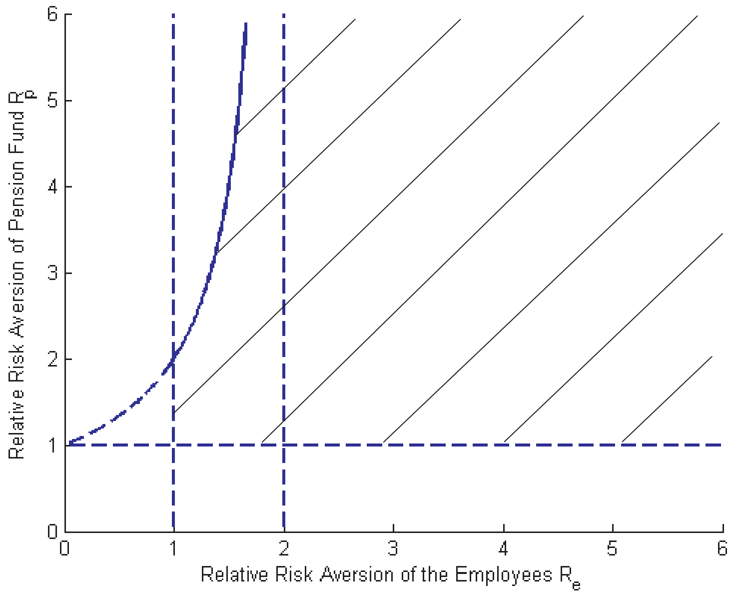
1. The restriction to linear attribution schemes leads to an optimal value of the participation rate α^* which exclusively depends on the relative risk aversion levels of the pension fund and the employees, respectively.

2. The Pareto-optimal portfolio (23) is more aggressive than the Merton portfolio if and only if

$$\begin{aligned} \frac{(R_e + R_p - 3)\,R_p + 2}{R_p^2 R_e} &> \frac{1}{R_p} \\ &\Leftrightarrow\ (R_p - 3)\,R_p + 2 > 0 \\ &\Leftrightarrow\ (R_p - 2)\,(R_p - 1) > 0 \\ &\Leftrightarrow\ R_p > 2. \end{aligned}$$

   Not surprisingly the question whether the optimal investment strategy is more aggressive compared to the Merton portfolio does not depend on the relative risk aversion of the employees. This stems

Figure 4: The shaded area displays the pairs of the relative risk aversion levels (R_e, R_p) which satisfy Assumption 5.2.



from our model setup where the pension fund chooses the investment rule for a given attribution scheme (α, a). Moreover, the comparison to the Merton investment rule naturally depends on the optimal participation rate α^* (indirect effect via R_p) but it does not depend on the premium a. This is of course a special case of the result in Remark 3.2.2.

□

The following proposition deals with the parameter sensitivity of the optimal participation rate and the Pareto-optimal investment strategy.

Proposition 5.6 *Let Assumption 5.2 be given.*

a) *Then, for the parameter sensitivities of the optimal participation one obtains:*

i)

$$\frac{\partial \alpha^*}{\partial R_p} > 0 \Leftrightarrow R_e > 2\left(\frac{R_p - 1}{R_p}\right)^2.$$

In particular, this implies

$$R_e \geq 2 \Rightarrow \frac{\partial \alpha^*}{\partial R_p} > 0.$$

ii)

$$\frac{\partial \alpha^*}{\partial R_e} = -\frac{R_p^2 (R_p - 1)}{[(R_e + R_p - 3) R_p + 2]^2} < 0.$$

b) *Then, for the parameter sensitivities of the Pareto-optimal investment strategy one obtains:*

i)

$$\frac{\partial \mathbf{x}^{Pa*}}{\partial R_p} = -\frac{(R_e - 3) R_p + 4}{R_p^3 R_e} \mathbf{V}^{-1} \pi.$$

Hence, increasing the relative risk aversion of the pension fund leads to a less aggressive Pareto-optimal investment strategy if and only if

$$R_e > 3 - \frac{4}{R_p}.$$

In particular, increasing the relative risk aversion of the pension fund leads to a less aggressive Pareto-optimal investment strategy if $R_p \leq 2$*.*

ii)

$$\frac{\partial \mathbf{x}^{Pa*}}{\partial R_e} = -\frac{(R_p - 2)(R_p - 1)}{R_p^2 R_e^2} \mathbf{V}^{-1}\pi.$$

Hence, increasing the relative risk aversion of the employees leads to a less aggressive Pareto-optimal investment strategy if and only if $R_p > 2$*.*

Proof:

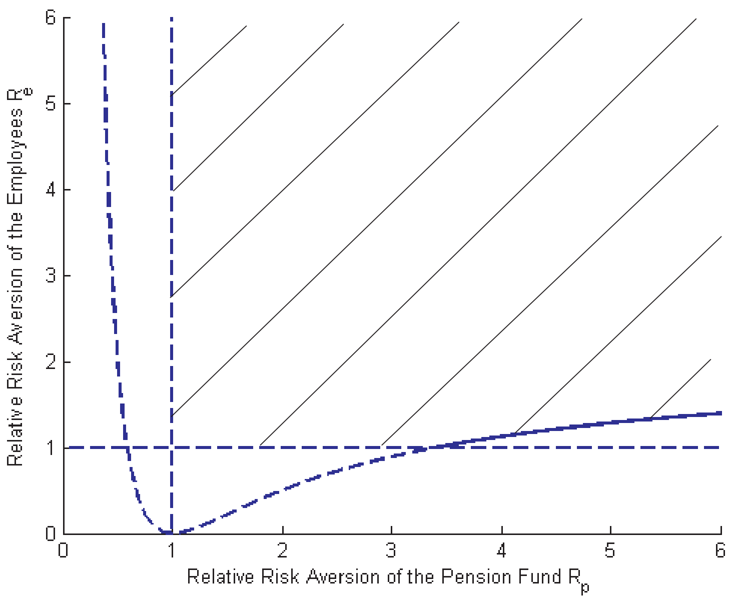
See Appendix A.3.

■

REMARK: 5.7

1. According to i) in part a) of Proposition 5.6 we naturally get a higher optimal participation rate the higher the relative risk aversion of the pension fund in most parameter settings. These parameter settings are displayed by the shaded area in Figure 5.

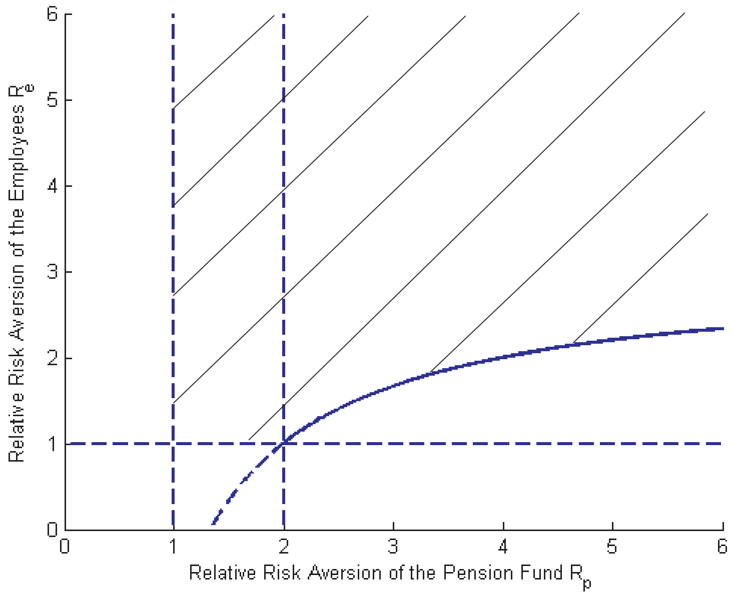
Figure 5: The shaded area displays the pairs of the relative risk aversion levels (R_p, R_e) for which an increase in the relative risk aversion of the pension fund naturally leads to a higher optimal participation rate.



2. According to ii) in part a) of Proposition 5.6 we always get a lower optimal participation rate the higher the relative risk aversion of the employees as expected.

3. According to i) in part b) of Proposition 5.6 we obtain the intuitive result that increasing the relative risk aversion of the pension fund leads to a less aggressive Pareto-optimal investment strategy in most parameter settings. These parameter settings are displayed by the shaded area in Figure 6.

Figure 6: The shaded area displays the pairs of the relative risk aversion levels (R_p, R_e) for which an increase in the relative risk aversion of the pension fund naturally leads to a less aggressive Pareto-optimal investment strategy.



□

6 Welfare Gain Measured by Certainty Equivalents

In this Section we want to measure the welfare gain which results from replacing an attribution scheme with no participation with an attribution scheme with optimal participation. The welfare gain will be measured on the basis of appropriate certainty equivalents. It will be shown that the welfare gain does not depend on the particular choice of the Pareto-optimal attribution scheme and the scheme with no participation.

6.1 Measures of Welfare

6.1.1 Welfare of the Employees

For the employees the certainty equivalent $X^c_{t,\tau}$ of the random variable $X_{t,\tau}$ is defined by

$$(1-R_e)^{-1}\left(\frac{X^c_{t,\tau}}{X_{t,0}}\right)^{1-R_e} = (1-R_e)^{-1}\,E\left[\left(\frac{X_{t,\tau}}{X_{t,0}}\right)^{1-R_e}\right]$$

or

$$\exp\left\{(1-R_e)\left(\ln X^c_{t,\tau} - \ln X_{t,0}\right)\right\} = E\left[\exp\left\{(1-R_e)\left(\ln X_{t,\tau} - \ln X_{t,0}\right)\right\}\right].$$

Since $X_{t,\tau}$ is lognormal[6] one obtains

$$\exp\left\{(1-R_e)\left(\ln X_{t,\tau}^c - \ln X_{t,0}\right)\right\} = \exp\left\{(1-R_e)\,E\left[\ln X_{t,\tau} - \ln X_{t,0}\right] + \frac{(1-R_e)^2}{2} Var\left[\ln X_{t,\tau} - \ln X_{t,0}\right]\right\}$$

or

$$\begin{aligned}\ln X_{t,\tau}^c - \ln X_{t,0} &= E\left[\ln X_{t,\tau} - \ln X_{t,0}\right] + \frac{1-R_e}{2} Var\left[\ln X_{t,\tau} - \ln X_{t,0}\right] \\ &= \left(\mu_e - \frac{\sigma_e^2}{2}\right)\tau + \frac{1-R_e}{2}\sigma_e^2\tau.\end{aligned}$$

The welfare of the employees is measured by

$$W_e(\mu_e, \sigma_e) = \frac{\ln X_{t,\tau}^c - \ln X_{t,0}}{\tau} = \mu_e - \frac{R_e}{2}\sigma_e^2. \tag{24}$$

Hence, the welfare of the employees is equal to the risk-adjusted rate of return on the employees' accounts.

6.1.2 Welfare of the Pension Fund

For the pension fund the certainty equivalent F_T^c of the random variable F_T is defined by

$$(1-R_p)^{-1}\left(\frac{F_T^c}{F_0}\right)^{1-R_p} = (1-R_p)^{-1} E\left[\left(\frac{F_T}{F_0}\right)^{1-R_p}\right]$$

or

$$\exp\left\{(1-R_p)\left(\ln F_T^c - \ln F_0\right)\right\} = E\left[\exp\left\{(1-R_p)\left(\ln F_T - \ln F_0\right)\right\}\right].$$

Since F_T is lognormal one obtains

$$\exp\left\{(1-R_p)\left(\ln F_T^c - \ln F_0\right)\right\} = \exp\left\{(1-R_p)\,E\left[\ln F_T - \ln F_0\right] + \frac{(1-R_p)^2}{2} Var\left[\ln F_T - \ln F_0\right]\right\}$$

or

$$\begin{aligned}\ln F_T^c - \ln F_0 &= E\left[\ln F_T - \ln F_0\right] + \frac{1-R_p}{2} Var\left[\ln F_T - \ln F_0\right] \\ &= \left(\mu_p - \frac{\sigma_p^2}{2}\right)T + \frac{1-R_p}{2}\sigma_p^2 T.\end{aligned}$$

The welfare of the pension fund is measured by

$$W_p(\mu_p, \sigma_p) = \frac{\ln F_T^c - \ln F_0}{T} = \mu_p - \frac{R_p}{2}\sigma_p^2. \tag{25}$$

Hence, the welfare of the pension fund is equal to the risk-adjusted growth rate of the funding ratio.

6.2 Welfare Gain

PROPOSITION 6.1 *Replacing an attribution scheme with no participation $(0, a_0)$ with an attribution scheme with optimal participation (α^*, a^*) leads to a welfare gain given by*

$$g = W_e(\mu_e, \sigma_e) + W_p(\mu_p, \sigma_p)\ \Big|_{(\alpha^*, a^*)} - W_e(\mu_e, \sigma_e) + W_p(\mu_p, \sigma_p)\ \Big|_{(0, a_0)} = \frac{(R_p - 1)^2}{2R_p^2 R_e}\pi^T \mathbf{V}^{-1}\pi. \tag{26}$$

[6]See (16).

Proof:

See Appendix A.4.

■

Remark: 6.2

1. The welfare gain naturally depends on the relative risk aversion of the pension fund and the employees, respectively. Moreover, it depends on the parameters of the financial market π and $\mathbf{V}$, respectively, as expected. However, the welfare gain does not depend on the particular choice of the attribution scheme with no participation $(0, a_0)$ and the attribution scheme with optimal participation (α^*, a^*), respectively, as the indifference curves of the pension fund and the employees only differ from each other by parallel shifts.

2. For $R_p = R_e = 3$ and $\pi^T \mathbf{V}^{-1} \pi = 6.25\%$ the welfare gain from optimal participation is

$$g = 0.463\%.$$

   Hence, under reasonable parameter values optimal participation allows for an increase in welfare which is quite substantial in the pension fund context.

3. We shortly describe an alternative way to measure the welfare gain that results from replacing an attribution scheme with no participation with a scheme with the optimal participation rate. As shown in Figure 7 we start with an attribution scheme $(0, a_0)$. This scheme will be compared with the attribution scheme $(\alpha^*, \widehat{a})$ leading to the same indirect utility level for the pension fund. In other words, $(0, a_0)$ and $(\alpha^*, \widehat{a})$ are on the same indifference curve of the pension fund, i.e.

$$a_0 - \widehat{a} = f_p(0) - f_p(\alpha^*) . \tag{27}$$

For the employees the attribution scheme $(\alpha^*, \widehat{a})$ leads to the same indirect utility level as the scheme $(0, a^*)$ with no participation, i.e.

$$a^* - \widehat{a} = f_e(0) - f_e(\alpha^*) . \tag{28}$$

Exploiting (27) and (28) the welfare gain measured by the risk-adjusted rate of return can be obtained as

$$g = a^* - a_0 = f_p(\alpha^*) - f_e(\alpha^*) - f_p(0) + f_e(0) . \tag{29}$$

This alternative approach using the change in the risk-adjusted rate of return leads to exactly the same expression for the welfare gain as the one we obtained in (26).

□

The following proposition discusses the parameter sensitivity of the welfare gain.

Proposition 6.3 *Let Assumption 5.2 be given. Then, for the parameter sensitivities of the welfare gain given in (26) one obtains:*

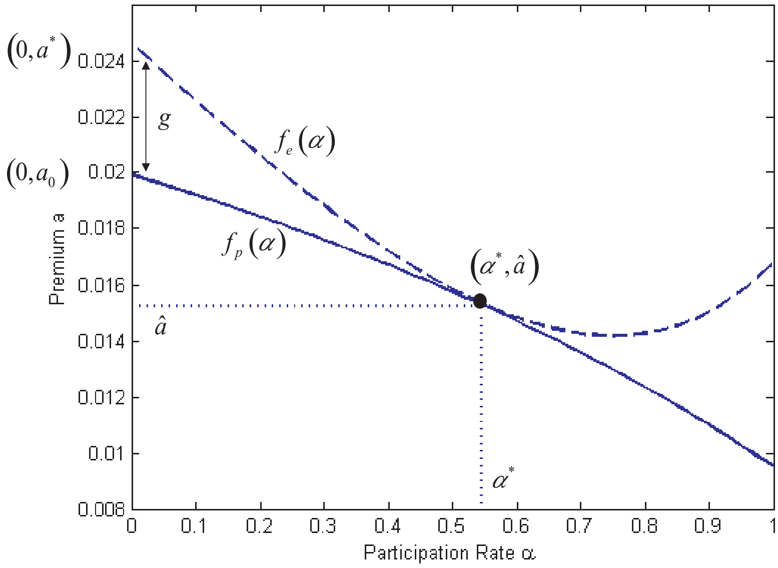
i)

$$\frac{\partial g}{\partial R_p} = \frac{R_p - 1}{R_p^3 R_e} \pi^T \mathbf{V}^{-1} \pi > 0.$$

ii)

$$\frac{\partial g}{\partial R_e} = -\frac{(R_p - 1)^2}{2 R_p^2 R_e^2} \pi^T \mathbf{V}^{-1} \pi < 0.$$

Figure 7: Illustration of the welfare gain measured by the increase in the risk-adjusted rate of return for $R_p = R_e = 3$, $r = 3\%$ and $\pi^T \mathbf{V}^{-1} \pi = 6.25\%$.



iii)

$$\frac{\partial g}{\partial R}\Big|_{R=R_e=R_p} = -\frac{1}{2}\frac{(R-3)(R-1)}{R^4}\pi^T \mathbf{V}^{-1} \pi.$$

Proof:

The proof is simple and therefore omitted for the sake of brevity.

■

Remark: 6.4

1. According to i) in Proposition 6.3 the welfare gain naturally increases in the relative risk aversion of the pension fund as risk sharing then becomes more valuable.

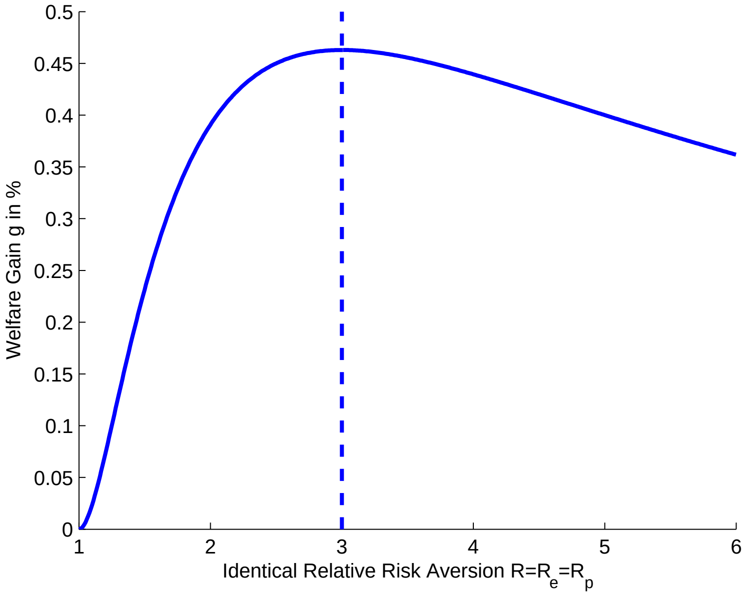
2. The parameter effect in ii) in Proposition 6.3 is less obvious at first sight. It results from the fact that the optimal participation rate is lower the higher the relative risk aversion of the employees.[7]

3. Finally, iii) in Proposition 6.3 shows the parameter sensitivity of the welfare gain for identical relative risk aversion levels of the pension fund and the employees, respectively. For $R < 3$ the effect dominates that a higher relative risk aversion makes risk sharing principly more valuable. For $R > 3$ however, the effect that a higher relative risk aversion decreases the aggressiveness of the Pareto-optimal investment strategy overcompensates the previsously mentioned effect. Figure 8 displays this result for $\pi^T \mathbf{V}^{-1} \pi = 6.25\%$.

□

[7]See ii) in part a) of Proposition 5.6.

Figure 8: The dependency of the welfare gain (in %) on identical relative risk aversion of the pension fund and the employees, respectively, for $\pi^T \mathbf{V}^{-1} \pi = 6.25\%$.



7 Conclusions

We analysed a continuous-time model where the employees participate in the pension fund's investment performance. We assumed that the pension fund derives utility from the funding ratio at some planning horizon and that the employees derive their utility from their savings accounts at the time they leave the pension fund. Moreover, for each linear attribution scheme the pension fund chooses the optimal investment strategy which was shown to be constant over time. Exploiting the optimal investment rule we derived the explicit funding ratio dynamics (geometric Brownian motion) and developed the indirect utility of the pension fund and the employees, respectively. We then showed that all Pareto-optima are characterised by the same optimal participation rate which naturally only depends on the relative risk aversion of the pension fund and the employees, respectively. Hence, the Pareto path is a vertical straight line. Exploiting the optimal participation rate we derived the Pareto-optimal investment strategy which is proportional to the optimal growth portfolio. Furthermore, we discussed the natural parameter sensitivities of the optimal participation rate and the Pareto-optimal investment strategy, respectively. Afterwards, we defined the welfare of the pension fund and the employees, respectively, by making use of appropriate certainty equivalents. Using these welfare definitions, we then derived the welfare gain that installs when replacing an attribution scheme with no participation with an attribution scheme with optimal participation. Alternatively, we could have measured the welfare gain by the increase in the risk-adjusted rate of return which naturally leads to exactly the same expression for the welfare gain. Finally, we showed that the welfare gain is substantial for realistic parameter values.

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A Appendix

A.1 Proof of Proposition 3.1

Proof:

The pension fund maximizes his utility defined in (5)

$$U_p\left(F_T\right)=\frac{E\left[F_T^{1-R_p}\right]}{1-R_p}$$

subject to the funding ratio evolution given in (4)

$$\frac{dF_t}{F_t}=\left[\left(1-\alpha\right)\mathbf{x}_t^T\pi-a+\left(\alpha^2-\alpha\right)\mathbf{x}_t^T\mathbf{V}\mathbf{x}_t\right]dt+\left(1-\alpha\right)\mathbf{x}_t^T\sigma\mathbf{dZ}_t.$$

The corresponding HJB-equation is

$$0=J_t+\max_{\{x_t\}\in A}\left\{J_FF_t\left[\left(1-\alpha\right)\mathbf{x}_t^T\pi-a+\left(\alpha^2-\alpha\right)\mathbf{x}_t^T\mathbf{V}\mathbf{x}_t\right]+0.5J_{FF}F_t^2\left(1-\alpha\right)^2\mathbf{x}_t^T\mathbf{V}\mathbf{x}_t\right\} \quad (30)$$

with the terminal condition

$$J\left(T,F_T\right)=\frac{F_T^{1-R_p}}{1-R_p}$$

where A denotes the set of admissible investment strategies.

We try the solution

$$J\left(t,F_t\right)=\frac{F_t^{1-R_p}}{1-R_p}$$

and get the following derivatives

$$\begin{aligned}J_t &= 0\\ J_F &= F_t^{-R_p}\\ J_{FF} &= -R_pF_t^{-R_p-1}.\end{aligned}$$

Inserting the above derivatives into the HJB-equation (30) yields

$$0=\max_{\{x_t\}\in A}\left\{F_t^{-R_p}F_t\left[\left(1-\alpha\right)\mathbf{x}_t^T\pi-a+\left(\alpha^2-\alpha\right)\mathbf{x}_t^T\mathbf{V}\mathbf{x}_t\right]-0.5R_pF_t^{-R_p-1}F_t^2\left(1-\alpha\right)^2\mathbf{x}_t^T\mathbf{V}\mathbf{x}_t\right\}.$$

The necessary and sufficient condition in the above maximisation is

$$0=F_t^{-R_p+1}\left[\left(1-\alpha\right)\pi+\left(\alpha^2-\alpha\right)2\mathbf{V}\mathbf{x}_t\right]-0.5R_pF_t^{-R_p+1}\left(1-\alpha\right)^2 2\mathbf{V}\mathbf{x}_t.$$

Solving for the optimal investment rule we obtain

$$\begin{aligned}-\left(1-\alpha\right)\pi &= 2\mathbf{V}\mathbf{x}_t\left[\left(\alpha^2-\alpha\right)-0.5R_p\left(1-\alpha\right)^2\right]\\ \mathbf{x}_t &= -\frac{1-\alpha}{2\left[\alpha\left(\alpha-1\right)-0.5R_p\left(1-\alpha\right)^2\right]}\mathbf{V}^{-1}\pi.\end{aligned}$$

Thus, the optimal investment strategy is given by

$$\mathbf{x}^*=\frac{1}{2\alpha+R_p\left(1-\alpha\right)}\mathbf{V}^{-1}\pi$$

which completes the proof.

■

A.2 Proof of Proposition 5.4

Proof:

a) We first prove that α^* given in (22) is the optimal participation rate.

Recalling (15) we have

$$a = f_p(\alpha) = 0.5\frac{1-\alpha}{2\alpha+R_p(1-\alpha)}\pi^T\mathbf{V}^{-1}\pi - C_p.$$

Differentiating $f_p(\alpha)$ with respect to α we get

$$\begin{aligned} f_p'(\alpha) &= 0.5\frac{-[2\alpha+R_p(1-\alpha)]-(1-\alpha)(2-R_p)}{[2\alpha+R_p(1-\alpha)]^2}\pi^T\mathbf{V}^{-1}\pi \\ &= -\frac{1}{[R_p+\alpha(2-R_p)]^2}\pi^T\mathbf{V}^{-1}\pi < 0. \end{aligned} \quad (31)$$

Recalling (20) we have

$$a = f_e(\alpha) = \frac{0.5R_e\alpha^2 - 2\alpha^2 - R_p\alpha(1-\alpha)}{[2\alpha+R_p(1-\alpha)]^2}\pi^T\mathbf{V}^{-1}\pi + C_e - r.$$

Calculating $f_e'(\alpha)$ yields

$$\begin{aligned} f_e'(\alpha) &= \frac{\left\{\begin{array}{c} [2\alpha+R_p(1-\alpha)]^2[R_e\alpha-4\alpha-R_p(1-\alpha)+R_p\alpha] \\ -2[2\alpha+R_p(1-\alpha)](2-R_p)\left[0.5R_e\alpha^2-2\alpha^2-R_p\alpha(1-\alpha)\right] \end{array}\right\}}{[2\alpha+R_p(1-\alpha)]^4}\pi^T\mathbf{V}^{-1}\pi \\ &= \frac{\left\{\begin{array}{c} [R_p+\alpha(2-R_p)][R_e\alpha-2\alpha(2-R_p)-R_p] \\ -(2-R_p)\alpha[R_e\alpha-2\alpha(2-R_p)-2R_p] \end{array}\right\}}{[2\alpha+R_p(1-\alpha)]^3}\pi^T\mathbf{V}^{-1}\pi. \end{aligned}$$

We next develop the numerator separately and obtain

$$\begin{aligned} & [R_p+\alpha(2-R_p)][R_e\alpha-2\alpha(2-R_p)-R_p]-(2-R_p)\alpha[R_e\alpha-2\alpha(2-R_p)-R_p]+R_p(2-R_p)\alpha \\ =\ & R_p[R_e\alpha-2\alpha(2-R_p)-R_p]+R_p(2-R_p)\alpha \\ =\ & R_p[R_e\alpha-\alpha(2-R_p)-R_p]. \end{aligned}$$

Hence, the derivative $f_e'(\alpha)$ is given by

$$f_e'(\alpha) = -R_p\frac{R_p+\alpha(2-R_p)-R_e\alpha}{[R_p+\alpha(2-R_p)]^3}\pi^T\mathbf{V}^{-1}\pi. \quad (32)$$

We next develop the necessary condition for Pareto-optima $f_p'(\alpha) - f_e'(\alpha) = 0$. Exploiting (31) and (32) gives

$$\begin{aligned} f_p'(\alpha) &= f_e'(\alpha) \\ -\frac{1}{[R_p+\alpha(2-R_p)]^2}\pi^T\mathbf{V}^{-1}\pi &= -\frac{R_p[R_p+\alpha(2-R_p)-R_e\alpha]}{[R_p+\alpha(2-R_p)]^3}\pi^T\mathbf{V}^{-1}\pi \\ -[R_p+\alpha(2-R_p)] &= R_pR_e\alpha - R_p[R_p+\alpha(2-R_p)] \\ R_p(R_p-1) &= \alpha\{R_pR_e+(1-R_p)(2-R_p)\}. \end{aligned}$$

Solving for α yields our claim in (22)

$$\alpha^* = \frac{R_p(R_p-1)}{(R_e+R_p-3)R_p+2}.$$

According to Assumption 5.2 we obviously have

$$\alpha^* = \frac{R_p (R_p - 1)}{R_p (R_p - 1) + R_p (R_e - 2) + 2} \in (0, 1) .$$

It remains to show that α^* represents a global maximum of $f_p(\alpha) - f_e(\alpha)$. Since $f_p(\alpha) - f_e(\alpha)$ is twice differentiable on $[0, 1]$ and since the equation $f'_p(\alpha) - f'_e(\alpha) = 0$ has the unique root α^* one only needs to show that $f''_p(\alpha^*) - f''_e(\alpha^*) < 0$. Exploiting (31) and (32) one obtains

$$\begin{aligned}
& f''_p(\alpha^*) - f''_e(\alpha^*) \\
= \; & \frac{2 (2 - R_p)}{[R_p + \alpha^* (2 - R_p)]^3} \pi^T \mathbf{V}^{-1} \pi \\
& - \frac{1}{[R_p + \alpha^* (2 - R_p)]^6} \Big(R_p [R_p + \alpha^* (2 - R_p)]^3 (R_p + R_e - 2) \\
& - 3 [R_p + \alpha^* (2 - R_p)]^2 (2 - R_p) R_p \{R_e \alpha^* - [R_p + \alpha^* (2 - R_p)]\} \Big) \pi^T \mathbf{V}^{-1} \pi .
\end{aligned}$$

Thus, we have

$$\begin{gathered}
f''_p(\alpha^*) - f''_e(\alpha^*) < 0 \\
\Leftrightarrow 2 (2 - R_p) [R_p + \alpha^* (2 - R_p)] - R_p (R_e + R_p - 2) [R_p + \alpha^* (2 - R_p)] \\
+ 3 (2 - R_p) R_p \{R_e \alpha^* - [R_p + \alpha^* (2 - R_p)]\} < 0 \\
\Leftrightarrow [R_p + \alpha^* (2 - R_p)] \{2 (2 - R_p) - R_p (R_e + R_p - 2) - 3 (2 - R_p) R_p\} + 3 (2 - R_p) R_p R_e \alpha^* < 0 .
\end{gathered}$$

Plugging the optimal participation rate (22) into the above yields

$$\begin{gathered}
\Leftrightarrow \{[(R_e + R_p - 3) R_p + 2] R_p + R_p (R_p - 1)(2 - R_p)\} \left\{2R_p^2 - 6R_p + 4 - R_p R_e\right\} + 3 (2 - R_p) R_p^2 (R_p - 1) R_e < 0 \\
\Leftrightarrow \left\{R_e R_p + R_p^2 - 3R_p + 2 - R_p^2 + 3R_p - 2\right\} \left\{2R_p^2 - 6R_p + 4 - R_p R_e\right\} + 3 (2 - R_p) R_p (R_p - 1) R_e < 0 \\
\Leftrightarrow R_e R_p \left(2R_p^2 - 6R_p + 4 - R_p R_e\right) + \left(-3R_p^2 + 9R_p - 6\right) R_p R_e < 0 \\
\Leftrightarrow -R_p^2 + 3R_p - 2 - R_p R_e < 0 \\
\Leftrightarrow R_p (R_p + R_e - 3) + 2 > 0 \\
\Leftrightarrow R_p (R_p - 1) - R_p (2 - R_e) + 2 > 0 .
\end{gathered}$$

According to Assumption 5.2 this last condition is satisfied.

b) Finally, we prove our claim in (23).

Plugging the optimal participation rate (22) into the general expression of the optimal investment rule (8) yields

$$\begin{aligned}
\mathbf{x}^{Pa*} &= \frac{1}{2\alpha^* + R_p (1 - \alpha^*)} \mathbf{V}^{-1} \pi \\
&= \frac{1}{2 \frac{R_p (R_p - 1)}{(R_e + R_p - 3) R_p + 2} + R_p \left(1 - \frac{R_p (R_p - 1)}{(R_e + R_p - 3) R_p + 2}\right)} \mathbf{V}^{-1} \pi \\
&= \frac{(R_e + R_p - 3) R_p + 2}{2R_p (R_p - 1) + R_p [(R_e + R_p - 3) R_p + 2 - R_p (R_p - 1)]} \mathbf{V}^{-1} \pi \\
&= \frac{(R_e + R_p - 3) R_p + 2}{R_p^2 R_e} \mathbf{V}^{-1} \pi
\end{aligned}$$

which completes the proof.

■

A.3 Proof of Proposition 5.6

Proof:

a) We first prove the parameter sensitivity of the optimal participation rate.

i) Differentiating the optimal participation rate given in (22) with respect to the relative risk aversion of the pension fund yields

$$\begin{aligned}
\frac{\partial \alpha^*}{\partial R_p} &= \frac{\left[\left(R_e+R_p-3\right)R_p+2\right]\left(2R_p-1\right)-\left(R_e+2R_p-3\right)R_p\left(R_p-1\right)}{\left[\left(R_e+R_p-3\right)R_p+2\right]^2} \\
&= \frac{\left(R_e+R_p-3\right)\left[R_p\left(2R_p-1\right)-R_p\left(R_p-1\right)\right]+2\left(2R_p-1\right)-R_p^2\left(R_p-1\right)}{\left[\left(R_e+R_p-3\right)R_p+2\right]^2} \\
&= \frac{\left(R_e+R_p-3\right)R_p^2-R_p^3+R_p^2+4R_p-2}{\left[\left(R_e+R_p-3\right)R_p+2\right]^2} \\
&= \frac{\left(R_e-2\right)R_p^2+4R_p-2}{\left[\left(R_e+R_p-3\right)R_p+2\right]^2}.
\end{aligned}$$

Concentrating on the sign of the above derivative we obtain

$$\begin{aligned}
\left(R_e-2\right)R_p^2+4R_p-2 &> 0 \\
R_eR_p^2 &> 2-4R_p+2R_p^2 \\
R_e &> \frac{2\left(1-2R_p+R_p^2\right)}{R_p^2} \\
R_e &> 2\left(\frac{R_p-1}{R_p}\right)^2.
\end{aligned}$$

The right hand side of the above inequality is obviously smaller than 2. Hence, one obtains $\frac{\partial \alpha^*}{\partial R_p} > 0$ if $R_e > 2$.

ii) Differentiating the optimal participation rate given in (22) with respect to the relative risk aversion of the employees immediately yields

$$\frac{\partial \alpha^*}{\partial R_e} = -\frac{R_p^2\left(R_p-1\right)}{\left[\left(R_e+R_p-3\right)R_p+2\right]^2} < 0.$$

b) Finally, we prove the parameter sensitivity of the Pareto-optimal investment strategy.

i) Differentiating the Pareto-optimal investment rule given in (23) with respect to the relative risk aversion of the pension fund gives

$$\begin{aligned}
\frac{\partial \mathbf{x}^{Pa*}}{\partial R_p} &= \frac{R_p^2R_e\left(R_e+2R_p-3\right)-2R_pR_e\left[\left(R_e+R_p-3\right)R_p+2\right]}{\left[R_p^2R_e\right]^2}\mathbf{V}^{-1}\pi \\
&= \frac{R_p^2R_e\left(R_e+R_p-3\right)+R_p^3R_e-\left(R_e+R_p-3\right)2R_p^2R_e-4R_pR_e}{R_p^4R_e^2}\mathbf{V}^{-1}\pi \\
&= \frac{R_p^3R_e-\left(R_e+R_p-3\right)R_p^2R_e-4R_pR_e}{R_p^4R_e^2}\mathbf{V}^{-1}\pi \\
&= \frac{-\left(R_e-3\right)R_p^2R_e-4R_pR_e}{R_p^4R_e^2}\mathbf{V}^{-1}\pi \\
&= -\frac{\left(R_e-3\right)R_p+4}{R_p^3R_e}\mathbf{V}^{-1}\pi.
\end{aligned}$$

Concentrating on the sign of the scalar in the above derivative we obtain

$$\begin{aligned}(R_e - 3) R_p + 4 &> 0 \\ R_e &> \frac{3R_p - 4}{R_p} = 3 - \frac{4}{R_p}.\end{aligned}$$

This inequality is clearly satisfied for $R_p \leq 2$.

ii) Differentiating the Pareto-optimal investment rule given in (23) with respect to the relative risk aversion of the employees yields

$$\begin{aligned}\frac{\partial \mathbf{x}^{Pa*}}{\partial R_e} &= \frac{R_p^2 R_e R_p - R_p^2 \left[(R_e + R_p - 3) R_p + 2 \right]}{\left[R_p^2 R_e \right]^2} \mathbf{V}^{-1} \pi \\ &= \frac{R_p R_e - (R_e + R_p - 3) R_p - 2}{R_p^2 R_e^2} \mathbf{V}^{-1} \pi \\ &= -\frac{R_p^2 - 3R_p + 2}{R_p^2 R_e^2} \mathbf{V}^{-1} \pi \\ &= -\frac{(R_p - 2)(R_p - 1)}{R_p^2 R_e^2} \mathbf{V}^{-1} \pi.\end{aligned}$$

■

A.4 Proof of Proposition 6.1

Proof:

Recalling (19) and (14) we have

$$\begin{aligned}W_e(\mu_e, \sigma_e) &= \mu_e - \frac{R_e}{2} \sigma_e^2 \\ &= r + a + \alpha \frac{2\alpha + R_p (1 - \alpha) - 0.5 R_e \alpha}{\left[2\alpha + R_p (1 - \alpha) \right]^2} \pi^T \mathbf{V}^{-1} \pi \qquad (33)\end{aligned}$$

and

$$\begin{aligned}W_p(\mu_p, \sigma_p) &= \mu_p - \frac{R_p}{2} \sigma_p^2 \\ &= \frac{0.5 (1 - \alpha)}{2\alpha + R_p (1 - \alpha)} \pi^T \mathbf{V}^{-1} \pi - a. \qquad (34)\end{aligned}$$

Thus, the welfare sum can be written as

$$\begin{aligned}& W_e(\mu_e, \sigma_e) + W_p(\mu_p, \sigma_p) \\ = \; & r + \frac{2\alpha^2 + R_p (1 - \alpha) \alpha - 0.5 R_e \alpha^2 + 0.5 (1 - \alpha) \left[2\alpha + R_p (1 - \alpha) \right]}{\left[2\alpha + R_p (1 - \alpha) \right]^2} \pi^T \mathbf{V}^{-1} \pi \\ = \; & r + \frac{2\alpha^2 - 0.5 R_e \alpha^2 + (1 - \alpha) \alpha + 0.5 (1 - \alpha) R_p (1 + \alpha)}{\left[2\alpha + R_p (1 - \alpha) \right]^2} \pi^T \mathbf{V}^{-1} \pi \\ = \; & r + \frac{-0.5 R_e \alpha^2 + (1 + \alpha) \left[\alpha + 0.5 (1 - \alpha) R_p \right]}{\left[2\alpha + R_p (1 - \alpha) \right]^2} \pi^T \mathbf{V}^{-1} \pi \\ = \; & r + \left\{ \frac{1 + \alpha}{2 \left[2\alpha + R_p (1 - \alpha) \right]} - \frac{R_e \alpha^2}{2 \left[2\alpha + R_p (1 - \alpha) \right]^2} \right\} \pi^T \mathbf{V}^{-1} \pi. \qquad (35)\end{aligned}$$

We first compute the sum of the welfare of the employees and the pension fund in the attribution scheme with no participation $(0, a_0)$ and obtain

$$W_e(\mu_e, \sigma_e) + W_p(\mu_p, \sigma_p) \Big|_{(0, a_0)} = r + \frac{1}{2R_p} \pi^T \mathbf{V}^{-1} \pi. \qquad (36)$$

We next compute the welfare sum in the attribution scheme with optimal participation (α^*, a^*)

$$W_e(\mu_e,\sigma_e)+W_p(\mu_p,\sigma_p)\underset{(\alpha^*,a^*)}{|}=r+\left\{\frac{1+\alpha^*}{2\left[2\alpha^*+R_p(1-\alpha^*)\right]}-\frac{R_e(\alpha^*)^2}{2\left[2\alpha^*+R_p(1-\alpha^*)\right]^2}\right\}\pi^T\mathbf{V}^{-1}\pi.$$

Plugging the optimal participation rate given in (22)

$$\alpha^*=\frac{R_p(R_p-1)}{(R_e+R_p-3)R_p+2}$$

into the above yields

$$\begin{aligned}
& W_e(\mu_e,\sigma_e)+W_p(\mu_p,\sigma_p)\underset{(\alpha^*,a^*)}{|} \\
= \;& r+\frac{1+\frac{R_p(R_p-1)}{(R_e+R_p-3)R_p+2}}{2\left[2\frac{R_p(R_p-1)}{(R_e+R_p-3)R_p+2}+R_p\left(1-\frac{R_p(R_p-1)}{(R_e+R_p-3)R_p+2}\right)\right]}\pi^T\mathbf{V}^{-1}\pi \\
& -\frac{R_e\left(\frac{R_p(R_p-1)}{(R_e+R_p-3)R_p+2}\right)^2}{2\left[2\frac{R_p(R_p-1)}{(R_e+R_p-3)R_p+2}+R_p\left(1-\frac{R_p(R_p-1)}{(R_e+R_p-3)R_p+2}\right)\right]^2}\pi^T\mathbf{V}^{-1}\pi \\
= \;& r+\frac{(R_e+R_p-3)R_p+2+R_p(R_p-1)}{2\left[2R_p(R_p-1)+R_p\left\{(R_e+R_p-3)R_p+2-R_p(R_p-1)\right\}\right]}\pi^T\mathbf{V}^{-1}\pi \\
& -\frac{R_e(R_p(R_p-1))^2}{2\left[2R_p(R_p-1)+R_p\left\{(R_e+R_p-3)R_p+2-R_p(R_p-1)\right\}\right]^2}\pi^T\mathbf{V}^{-1}\pi \\
= \;& r+\frac{(R_e+2R_p-4)R_p+2}{2\left[2R_p(R_p-1)+(R_e+R_p-3)R_p^2+2R_p-R_p^2(R_p-1)\right]}\pi^T\mathbf{V}^{-1}\pi \\
& -\frac{R_eR_p^2(R_p-1)^2}{2\left[2R_p(R_p-1)+(R_e+R_p-3)R_p^2+2R_p-R_p^2(R_p-1)\right]^2}\pi^T\mathbf{V}^{-1}\pi \\
= \;& r+\left\{\frac{(R_e+2R_p-4)R_p+2}{2\left[(R_e+R_p)R_p^2-R_p^3\right]}-\frac{R_eR_p^2(R_p-1)^2}{2\left[(R_e+R_p)R_p^2-R_p^3\right]^2}\right\}\pi^T\mathbf{V}^{-1}\pi \\
= \;& r+\left\{\frac{(R_e+2R_p-4)R_p+2}{2R_eR_p^2}-\frac{R_eR_p^2(R_p-1)^2}{2\left[R_eR_p^2\right]^2}\right\}\pi^T\mathbf{V}^{-1}\pi \\
= \;& r+\frac{(R_e+2R_p-4)R_p+2-R_p^2+2R_p-1}{2R_eR_p^2}\pi^T\mathbf{V}^{-1}\pi \\
= \;& r+\frac{(R_e+R_p-2)R_p+1}{2R_eR_p^2}\pi^T\mathbf{V}^{-1}\pi. \qquad (37)
\end{aligned}$$

Replacing a scheme with no participation $(0, a_0)$ with a scheme with a Pareto-optimal participation (α^*, a^*) therefore leads to the following change in welfare

$$\begin{aligned}
& W_e(\mu_e,\sigma_e)+W_p(\mu_p,\sigma_p)\underset{(\alpha^*,a^*)}{|}-W_e(\mu_e,\sigma_e)+W_p(\mu_p,\sigma_p)\underset{(0,a_0)}{|} \\
= \;& r+\frac{(R_e+R_p-2)R_p+1}{2R_eR_p^2}\pi^T\mathbf{V}^{-1}\pi-\left(r+\frac{1}{2R_p}\pi^T\mathbf{V}^{-1}\pi\right) \\
= \;& \frac{(R_e+R_p-2)R_p+1-R_eR_p}{2R_eR_p^2}\pi^T\mathbf{V}^{-1}\pi \\
= \;& \frac{(R_p-1)^2}{2R_eR_p^2}\pi^T\mathbf{V}^{-1}\pi. \qquad (38)
\end{aligned}$$

Finally, we note that the above derivation of the welfare gain does not depend on the particular choice of the Pareto-optimal attribution scheme and the scheme with no participation. ■