

Investment Policies for Defined-Contribution Pension Funds

Heinz H. Müller, Roger T. Baumann*

University of St. Gallen, Bodanstrasse 6, 9000 St. Gallen, Switzerland

Abstract

Dynamic optimal investment policies are derived for a defined-contribution pension plan under expected utility maximization and under minimization of the shortfall probability. It is assumed that the return attributed by the fund on the accrued retirement benefits of employees may depend on the funding ratio and partly on the investment performance of the fund. The resulting investment policies are compared with the reference case where the attributed return is assumed to be constant. Under expected utility maximization the investment strategy becomes more aggressive than in the reference case if the attributed return increases with the funding ratio. Higher net contributions lead to a more aggressive investment strategy. However, participation of employees in the investment performance of the fund has an ambiguous effect on the investment policy. Under expected utility maximization the funding ratio is lognormally distributed and asymptotically well behaved. For the minimization of the shortfall probability a technique developed by Pestien and Sudderth (1985) is used. Formulas for the shortfall probability are derived and discussed.

Keywords: Portfolio choice; Pension finance; Defined-contribution pension fund; Shortfall probability.

1 Introduction

A central issue in pension finance is the determination of an optimal investment policy. In order to determine an optimal investment policy some information on the structure of the fund is needed. In particular, one has to distinguish between defined-benefit and defined-contribution plans. Since for a defined-benefit plan benefits are defined in advance the plan sponsor and the employees have to adjust contributions. Due to the fact that the plan sponsor has to bear substantial financial and actuarial risk and since it is difficult to value the accrued retirement benefits for employees changing their plan sponsor, this type of pension plan has become less popular in the recent past. In the literature the optimal investment policy for defined benefit plans is treated by Haberman and Sung (1994), Boulier et al. (1995), Cairns (2000) or Josa-Fombellida and Rincón-Zapatero (2006) among others.

*E-mail addresses: heinz.mueller@unisg.ch, roger.baumann@unisg.ch.

There exist various versions of defined-contribution plans. For a defined-contribution plan in the narrow sense the accrued retirement benefits of employees are invested and the positive or negative fund return is fully attributed to employee accounts. At retirement each employee obtains the final amount from his account as a cash payment, respectively as a corresponding pension. Moreover, employees may be allowed to choose individual investment policies (e.g. 401k-plans in the US, Swedish pension funds). The pension fund bears no financial risk at all and acts only as a broker between employees and financial and life insurance markets. Hence employees may be considered as individual investors and their optimal investment strategies can be derived with standard methods. This kind of problem was analyzed by several authors. Merton models with state variables (see Merton (1969)) and in particular Adler and Dumas (1983) deal with this topic in continuous time. In discrete time there are articles by Solnik (1978), Wise (1984a), Wise (1984b), Wilkie (1985), Sharpe and Tint (1990), Keel and Müller (1995), Leippold et al. (2004).

On the other hand, there exist defined-contribution plans where the plan sponsor bears substantial financial risk (e.g. mandatory part of Swiss pension plans) in order to protect employees against adverse movements in financial markets. For these plans the financial situation can be represented by the funding ratio, which is the ratio of the value of assets to the present value of net obligations. Of course, to assure equal treatment, in such a plan employees cannot choose individual investment policies. But in case of declining financial markets by decreasing the funding ratio the fund can protect to some extent the accrued benefits of retiring employees. During prospering markets high fund returns can be used to raise a low funding ratio. Baumann and Müller (2008) studied a model where the return attributed to the individual accounts of employees increases with the funding ratio but does not depend on the fund return. Such a plan-policy which increases the funding ratio if markets perform well and decreases it in case of poor market performance leads to an intergenerational transfer of financial risk among employees. Baumann and Müller (2008) show that in such a framework the fund can choose investment policies such that all employees would be worse off, if they acted as individual investors.

In this paper optimal investment strategies for defined-contribution plans will be derived. Similar to the articles by Browne (1999), Denzler et al. (2001), Müller and Baumann (2006) the funding ratio is in the focus of the optimization. But in contrast to these articles, as in the article by Baumann and Müller (2008) the liability is endogenous in a model where employees are partly protected against adverse market movements. It will be assumed that the return attributed to employee accounts depends as well on the funding ratio as on the funds return which is an extension to the model of Baumann and Müller (2008). First, optimal investment policies will be derived for the case where the expected utility of the funding ratio at the end of a planning horizon has to be maximized. The impact of the partial protection on the investment policy and the dynamics of the funding ratio will be discussed in detail. Thereafter, we shall analyze investment strategies minimizing the shortfall probability. A shortfall occurs if the funding ratio hits a prespecified low level before it attains some target level. The analysis can either be based

on Browne (1997) or on Pestien and Sudderth (1985). It turns out that the optimal investment policy is highly sensitive to the funding ratio. Finally, the model is extended in order to study the effect of non-zero net contributions on investment policies and on the dynamics of the funding ratio.

The paper is structured as follows. In section 2 the model is presented and the different model specifications are discussed. Section 3 deals with optimal investment policies under expected utility maximization. In section 4 some results by Pestien and Sudderth (1985) are summarized and applied to the shortfall minimization problem in our model. In section 5 the model is extended to cover non-zero net contributions to the pension fund. The main results are summarized in section 6.

2 Model Characteristics

In the sense of an overlapping generation model employees continuously enter the plan, pay their contributions and retire after a fixed time period. The fund continuously attributes a return on their accrued retirement benefits, which are paid out at retirement.

The model is focussed on the fund, which is modelled as follows:

- A_t : value of assets at time t
- L_t : value of liabilities at time t
- C_t : net contributions¹ at time t
- $F_t = \frac{A_t}{L_t}$: funding ratio at time t

There are $N + 1$ investment opportunities. The price process of the riskless investment opportunity $i = 0$ is given by

$$\frac{dS_{0,t}}{S_{0,t}} = rdt,$$

The price processes of the risky investment opportunities $i = 1, \dots, N$ are given by geometric Brownian motions, i.e.

$$\left(\frac{dS_{i,t}}{S_{i,t}} \right)_{i=1,\dots,N} = (r + \boldsymbol{\pi}_i)_{i=1,\dots,N} dt + \sigma d\mathbf{Z}_t, \quad (1)$$

where \mathbf{Z}_t denotes the N -dimensional standard Brownian Motion, σ is a regular matrix, $\boldsymbol{\pi}^\top = (\boldsymbol{\pi}_1, \dots, \boldsymbol{\pi}_N)$ denotes the vector of risk premia and $V = \sigma\sigma^\top$ denotes the covariance matrix.

The fund chooses an investment policy $\mathbf{x}_t \in \mathbb{R}^N$, $t \geq 0$, in order to invest A_t . This leads to

$$dA_t = A_t(r + \mathbf{x}_t^\top \boldsymbol{\pi}) dt + C_t dt + A_t \mathbf{x}_t^\top \sigma d\mathbf{Z}_t. \quad (2)$$

The return attributed by the fund to the accrued retirement benefits of the employees is given by

$$dR_t = [r + (1 - \alpha)g(F_t) + \alpha \mathbf{x}_t^\top \boldsymbol{\pi}] dt + \alpha \mathbf{x}_t^\top \sigma d\mathbf{Z}_t, \quad 0 \leq \alpha < 1. \quad (3)$$

For $\alpha = 0$ employees bear no short run financial risk and the return is given by $dR_t = (r + g(F_t)) dt$. The function $g(\cdot)$ is assumed to be increasing. Hence, $\alpha = 0$ represents the extreme case where the return on employee accounts is an increasing function of the funding ratio but does not directly depend on investment performance.

For $0 < \alpha < 1$ the return is partly related to the investment performance and partly to the funding ratio.

The dynamics of the fund liabilities are given by

$$dL_t = L_t dR_t + C_t dt$$

or

$$dL_t = [L_t (r + (1 - \alpha)g(F_t) + \alpha \mathbf{x}_t^\top \boldsymbol{\pi}) + C_t] dt + \alpha L_t \mathbf{x}_t^\top \sigma d\mathbf{Z}_t. \quad (4)$$

¹Contributions of employees minus retirement benefits.

From

$$dF_t = d\left(\frac{A_t}{L_t}\right) = \frac{dA_t}{L_t} - \frac{A_t}{L_t^2}dL_t + \frac{A_t}{L_t^3}(dL_t)^2 - \frac{1}{L_t^2}(dA_t)(dL_t)$$

one obtains

$$\frac{dF_t}{F_t} = \frac{dA_t}{A_t} - \frac{dL_t}{L_t} + \left(\frac{dL_t}{L_t}\right)^2 - \left(\frac{dA_t}{A_t}\right)\left(\frac{dL_t}{L_t}\right)$$

or

$$\frac{dF_t}{F_t} = \left[(1-\alpha)(\mathbf{x}_t^\top \boldsymbol{\pi} - g(F_t)) + \frac{C_t}{A_t} - \frac{C_t}{L_t} - \alpha(1-\alpha)\mathbf{x}_t^\top V\mathbf{x}_t \right] dt + (1-\alpha)\mathbf{x}_t^\top \sigma d\mathbf{Z}_t. \quad (5)$$

Remark As expected a positive participation rate α reduces the volatility of the funding ratio. The negative component $-\alpha(1-\alpha)\mathbf{x}_t^\top V\mathbf{x}_t$ in the drift term results from the fact that asset and liability returns become positively correlated under $\alpha > 0$. Due to those effects the participation rate will have an ambiguous effect on the investment policy.

Model Specifications

In this paper three different model specifications will be analyzed.

First, as a reference case we shall deal with the specification

$$\mathbf{A.1:} \quad \alpha = 0, g(F_t) \equiv a, C_t \equiv 0.$$

Under A.1 the dynamics of the funding ratio are given by

$$\frac{dF_t}{F_t} = (\mathbf{x}_t^\top \boldsymbol{\pi} - a) dt + \mathbf{x}_t^\top \sigma d\mathbf{Z}_t. \quad (6)$$

Comment on A.1:

The rate of return attributed to the employees is constant, i.e. it depends neither on the funding ratio nor on the performance. Under a constant relative risk aversion with respect to the funding ratio the Merton portfolio represents the optimal investment strategy.

The second case we shall deal with is characterized by

$$\mathbf{A.2:} \quad 0 \leq \alpha < 1, g(F_t) = k \ln\left(\frac{F_t}{\bar{F}}\right), k > 0, C_t \equiv 0.$$

Under A.2 the dynamics of the funding ratio are given by

$$\frac{dF_t}{F_t} = (1-\alpha)(\mathbf{x}_t^\top \boldsymbol{\pi} - k(\ln F_t - \ln \bar{F}) - \alpha\mathbf{x}_t^\top V\mathbf{x}_t) dt + (1-\alpha)\mathbf{x}_t^\top \sigma d\mathbf{Z}_t. \quad (7)$$

Comment on A.2

For $\alpha = 0$ the return on employee accounts is

$$dR_t = \left(r + k \cdot \ln\left(\frac{F_t}{\bar{F}}\right) \right) dt.$$

\bar{F} is a critical level for the funding ratio. Below this level employees get less than the riskfree rate on their accounts. Assumption A.2 will allow to obtain explicit solutions for the optimal investment policy. Since $\ln\left(\frac{F_t}{\bar{F}}\right)$ may be considered as a measure for over- or underfunding of a pension fund assumption

A.2 is quite natural. Under A.2 the return on employee accounts increases with the funding ratio. This leads to an intertemporal risk transfer (see Baumann and Müller (2008)).

Finally we shall analyze the case

A.3: $0 \leq \alpha < 1, g(F_t) = k \ln \left(\frac{F_t}{\bar{F}} \right), k > 0, \frac{C_t}{A_t} - \frac{C_t}{L_t} = -c \ln F_t, c \in \mathbb{R}.$

Under A.3 the dynamics of the funding ratio are given by

$$\frac{dF_t}{F_t} = [(1 - \alpha) \mathbf{x}_t^T \boldsymbol{\pi} - ((1 - \alpha)k + c) \ln F_t + (1 - \alpha)k \ln \bar{F} - \alpha(1 - \alpha) \mathbf{x}_t^T V \mathbf{x}_t] dt + (1 - \alpha) \mathbf{x}_t^T \sigma d\mathbf{Z}_t. \quad (8)$$

Comment on A.3

Under this assumption some analytical results on the impact of positive or negative net contributions on the optimal investment policy and on the dynamics of the funding ratio can be obtained. Assumption A.3 takes into account that positive net contributions draw the funding ratio towards one.

3 Expected utility maximization

In this section the expected utility of the funding ratio F_T at a planning horizon T is maximized under the model specifications A.1 and A.2. The analysis under model specification A.3 is deferred to section 5. The objective function of the fund is given by

$$E \left[(1 - R)^{-1} F_T^{1-R} \right], R > 1$$

Proposition 1 *The investment policy $x_t, 0 \leq t \leq T$, maximizing*

$$E \left[(1 - R)^{-1} F_T^{1-R} \right]$$

is given by

1)

$$\mathbf{x}_t = \frac{1}{R} V^{-1} \boldsymbol{\pi} \quad \text{under A.1} \quad (9)$$

2)

$$\mathbf{x}_t = \frac{1}{1 + \alpha + (1 - \alpha)(R - 1)e^{(1 - \alpha)k(t - T)}} V^{-1} \boldsymbol{\pi} \quad \text{under A.2.} \quad (10)$$

Proof. Part 1 is well known.

Part 2 follows from proposition 7. ■

Comment

Under A.1 the Merton portfolio results. Under A.2 the investment policy is more aggressive the farther away one is from the investment horizon T . At the investment horizon one obtains $\mathbf{x}_T = \frac{1}{2\alpha + (1 - \alpha)R}$ which is still more aggressive than the Merton portfolio if $R > 2$. For $\alpha = 0$ the investment strategy under A.2 is more aggressive than the Merton portfolio for $R > 1$. This result is not surprising since the model specification A.2 corresponds to an intertemporal risk transfer. Moreover, the impact of the participation rate α on the investment policy is in fact ambiguous.

Under A.2 one obtains from (8)

$$d(\ln(F_t)) = \left[(1 - \alpha) (\mathbf{x}_t^\top \boldsymbol{\pi} - k \ln F_t + k \ln \bar{F}) - \frac{1}{2} (1 - \alpha^2) \mathbf{x}_t^\top V \mathbf{x}_t \right] dt + (1 - \alpha) \mathbf{x}_t^\top \sigma d\mathbf{Z}_t. \quad (11)$$

Substituting $Y_t = \ln(F_t)$ and using (10) leads to the Ornstein Uhlenbeck process

$$\begin{aligned} dY_t = & \left[k(1 - \alpha) (\ln \bar{F} - Y_t) + \frac{(1 - \alpha) \boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}}{1 + \alpha + (1 - \alpha)(R - 1) e^{(1 - \alpha)k(t - T)}} \right. \\ & \left. - \frac{\frac{1}{2} (1 - \alpha^2) \boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}}{[1 + \alpha + (1 - \alpha)(R - 1) e^{(1 - \alpha)k(t - T)}]^2} \right] dt \\ & + \frac{(1 - \alpha) (\boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi})^{0.5}}{1 + \alpha + (1 - \alpha)(R - 1) e^{(1 - \alpha)k(t - T)}} dZ_t^*, \end{aligned} \quad (12)$$

where Z_t is a one-dimensional standard Brownian motion.

Proposition 2

The solution $Y_t = \ln(F_t)$ of (12) is Gaussian with

$$\begin{aligned} E(Y_t) = & E(Y_0) e^{-k(1 - \alpha)t} + \left(1 - e^{-k(1 - \alpha)t}\right) \ln \bar{F} \\ & + \frac{\boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi} e^{-k(1 - \alpha)(t - T)}}{2k(1 - \alpha)(R - 1)} \left[2 \ln \left(\frac{1 + \alpha + (1 - \alpha)(R - 1) e^{k(1 - \alpha)(t - T)}}{1 + \alpha + (1 - \alpha)(R - 1) e^{-k(1 - \alpha)T}} \right) \right. \\ & \left. + \frac{1 + \alpha}{1 + \alpha + (1 - \alpha)(R - 1) e^{k(1 - \alpha)(t - T)}} - \frac{1 + \alpha}{1 + \alpha + (1 - \alpha)(R - 1) e^{-k(1 - \alpha)T}} \right] \end{aligned} \quad (13)$$

$$\begin{aligned} Var(Y_t) = & e^{-2k(1 - \alpha)t} Var(Y_0) \\ & + \frac{\boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi} e^{2(1 - \alpha)k(T - t)}}{k(1 - \alpha)(R - 1)^2} \left[\frac{1 + \alpha}{1 + \alpha + (1 - \alpha)(R - 1) e^{(1 - \alpha)k(t - T)}} - \frac{1 + \alpha}{1 + \alpha + (1 - \alpha)(R - 1) e^{-k(1 - \alpha)T}} \right. \\ & \left. + \ln \left(\frac{1 + \alpha + (1 - \alpha)(R - 1) e^{k(1 - \alpha)(t - T)}}{1 + \alpha + (1 - \alpha)(R - 1) e^{-k(1 - \alpha)T}} \right) \right]. \end{aligned} \quad (14)$$

Proof. Follows from proposition 8. ■

$E(Y_t)$ and $Var(Y_t)$ in proposition 2 are not easy to interpret. Therefore, some properties of the funding ratio at the investment horizon T are summarized in the next proposition.

Proposition 3

1) $E(Y_T)$ and $Var(Y_T)$ are strictly decreasing in R .

2)

$$\lim_{T \rightarrow \infty} E(Y_T) > \ln \bar{F} \quad (15)$$

3)

$$\lim_{T \rightarrow \infty} E(Y_T) = \ln \bar{F} + \frac{\boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}}{k(1 - \alpha)(R - 1)} \ln \left(\frac{1 + \alpha + (1 - \alpha)(R - 1)}{1 + \alpha} \right) - \frac{\boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}}{2k[1 + \alpha + (1 - \alpha)(R - 1)]} \quad (16)$$

$$\lim_{T \rightarrow \infty} Var(Y_T) = \frac{\boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}}{k(1 - \alpha)(R - 1)} \left[\frac{1}{R - 1} \ln \left(\frac{1 + \alpha + (1 - \alpha)(R - 1)}{1 + \alpha} \right) - \frac{1 - \alpha}{1 + \alpha + (1 - \alpha)(R - 1)} \right] \quad (17)$$

4) For $\alpha = 0$:

$$\lim_{T \rightarrow \infty} E(Y_T) = \ln \bar{F} + \frac{\boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}}{k} \left(\frac{\ln R}{R-1} - \frac{1}{2R} \right) \quad (18)$$

$$\lim_{T \rightarrow \infty} Var(Y_T) = \frac{\boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}}{k(R-1)} \left[\frac{\ln R}{R-1} - \frac{1}{R} \right] \quad (19)$$

5) For $\alpha \rightarrow 1$:

$$\lim_{T \rightarrow \infty} E(Y_T) \rightarrow \ln \bar{F} + \frac{\boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}}{4k} \quad (20)$$

$$\lim_{T \rightarrow \infty} Var(Y_T) \rightarrow 0. \quad (21)$$

Proof. Follows from proposition 9. ■

Comments

ad 1) Since

$$E \left[(1-R)^{-1} F_T^{1-R} \right] = E \left[(1-R)^{-1} e^{(1-R) \ln F_T} \right] \quad (22)$$

$$= (1-R)^{-1} e^{(1-R)E(Y_T) + \frac{1}{2}(1-R)^2 Var(Y_T)} \quad (23)$$

holds, expected utility maximization is equivalent to mean variance optimization with respect to Y_T .

Therefore, the monotonicity of $E(Y_T)$, $Var(Y_T)$ is not surprising.

ad 2) and 3)

According to (15), (16), (17) the stochastic funding ratio F_T is asymptotically well behaved.

ad 4) and 5)

A comparison of (18) and (20) for different levels of relative risk aversion R shows that the impact of changes in the employee's participation rate α on $\lim_{T \rightarrow \infty} E(Y_T)$ is ambiguous. This ambiguity is also shown in figure 1 where

$$\begin{aligned} f(\alpha) &= \frac{k}{\boldsymbol{\pi}^\top V^{-1} \boldsymbol{\pi}} \left(\lim_{T \rightarrow \infty} E(Y_T) - \ln \bar{F} \right) \\ &= \frac{1}{(1-\alpha)(R-1)} \ln \left(\frac{1+\alpha+(1-\alpha)(R-1)}{1+\alpha} \right) - \frac{1}{2[1+\alpha+(1-\alpha)(R-1)]} \end{aligned} \quad (24)$$

is illustrated for different levels of R .

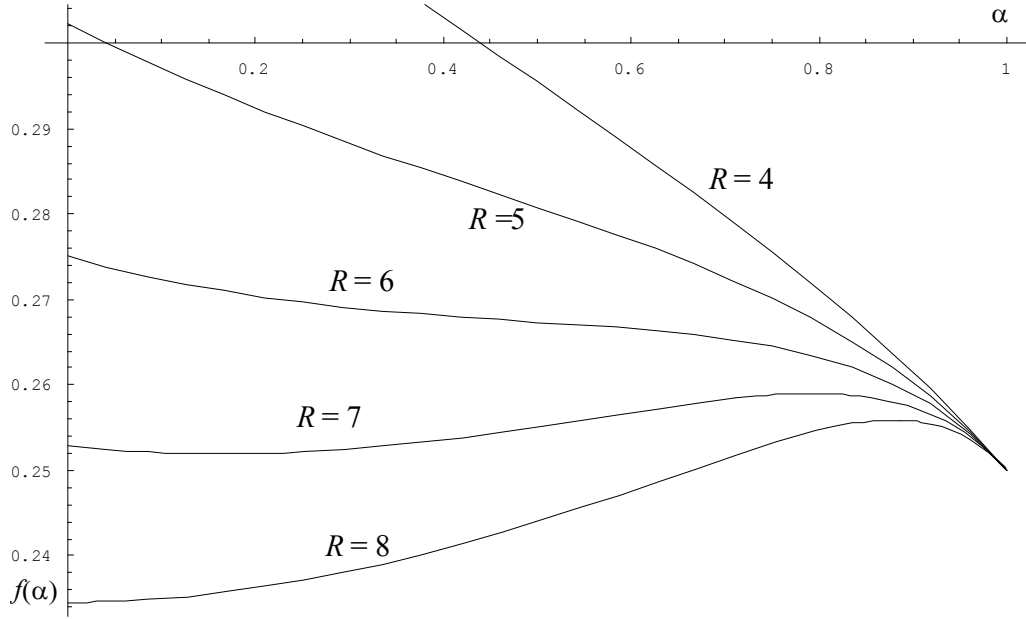


Figure 1: Impact of α on $\lim_{T \rightarrow \infty} E(Y_T)$.

Hence expected utility maximization leads to the following main results:

- For $R > 2$ the investment policy is more aggressive under model specification A.2 than under A.1.
- The logarithm of the funding ratio $\ln F_t$ follows a Ornstein Uhlenbeck process and is therefore Gaussian.
- The impact of the participation rate α on the investment policy and on $\lim_{T \rightarrow \infty} E(\ln F_T)$ is ambiguous.

4 Minimization of the Shortfall Probability

4.1 Shortfall Minimization Problem

Under the model specifications A.1, A.2 and an investment policy \mathbf{x}_t the dynamics of the funding ratio F_t are given by (6), respectively by (7). A shortfall occurs if the funding ratio hits a low level F_{\min} before attaining a target level F_{\max} .

More formally, it is assumed that the stochastic process F_t stops at τ_{\min} if $F_{\tau_{\min}} = F_{\min}$, respectively at τ_{\max} if $F_{\tau_{\max}} = F_{\max}$. The shortfall probability is denoted by $\text{Prob}[\tau_{\min} < \tau_{\max}]$.

Hence, the shortfall minimizing problem can be formulated as follows. Starting at an initial funding ratio $F_0 \in (F_{\min}, F_{\max})$ one has to find an investment policy \mathbf{x}_t such that for the stochastic process

$$d \ln F_t = \left(\mathbf{x}_t^T \boldsymbol{\pi} - a - \frac{1}{2} \mathbf{x}_t^T V \mathbf{x}_t \right) dt + \mathbf{x}_t^T \sigma d\mathbf{Z}_t \quad (25)$$

respectively

$$d \ln F_t = (1 - \alpha) \left(\mathbf{x}_t^T \boldsymbol{\pi} - k \ln F_t + k \ln \bar{F} - \frac{1}{2} (1 + \alpha) \mathbf{x}_t^T V \mathbf{x}_t \right) dt + (1 - \alpha) \mathbf{x}_t^T \sigma d\mathbf{Z}_t \quad (26)$$

the shortfall probability

$$\text{Prob}[\tau_{\min} < \tau_{\max}]$$

is minimized.

Shortfall problems are discussed in detail by Browne (1997).

However, we follow Pestien and Sudderth (1985) whose results can be directly applied to our model. The results of Pestien and Sudderth (1985) needed for our shortfall problem are shortly summarized in the next subsection.

4.2 Control of a Diffusion to a Goal

Let $X_t \in \mathcal{E}$ be stochastic processes with right-continuous paths and left limits satisfying the stochastic differential equation

$$dX_t = \mu(X_t) dt + \sigma(X_t) dZ_t, \quad X_0 \in (a, b), \quad (\mu(x), \sigma(x)) \in C(x), \quad x \in [a, b], \quad (27)$$

Z_t is a one-dimensional standard Brownian motion.

The control sets $C(x)$, $x \in [a, b]$ are non-empty and bounded, $C(x) \in \mathbb{R} \times [0, \infty)$ and $C(a) = C(b) = \{(0, 0)\}$. Hence, the processes X_t are absorbed at a and b .

The assumptions A, B in Pestien and Sudderth (1985), pp. 604, 605 can be replaced by

Assumption A*

$\mu_0(x), \sigma_0(x), x \in (a, b)$ with

$$(\mu_0(x), \sigma_0(x)) = \arg \max \left\{ \frac{\mu(x)}{\sigma^2(x)} \mid (\mu(x), \sigma(x)) \in C(x) \right\} \quad (28)$$

are well defined, bounded, Borel-measurable functions with $\inf_{x \in (a, b)} \sigma_0(x) > 0$.

Define the stochastic process X_t^* by

$$dX_t^* = \mu_0(X_t^*) dt + \sigma_0(X_t^*) dZ_t, \quad X_0^* = x.$$

The probability that X_t^* attains b is given by

$$\text{Prob}[\tau^* < \infty]$$

where

$$\tau^* := \inf \{t \mid X_t^* = b\}.$$

As a special case of the results by Pestien and Sudderth (1985), pp. 604-605 one obtains:

Theorem 4

Assumption A implies*

1.

$$\text{Prob}[\tau^* < \infty] = \frac{\int_a^x \xi(v) dv}{\int_a^x \xi(v) dv} \quad (29)$$

where $\xi(v) = \exp\left\{-2\int_a^v \frac{\mu_0(z)}{\sigma_0^2(z)} dz\right\}$.

2. Among all stochastic processes $X_t \in \mathcal{E}$, $X_0 = x$ the process X_t^* maximizes the probability to attain b .

4.3 Shortfall Minimizing Investment Policy

Now the method of Pestien and Sudderth (1985) can be applied by putting

$$X_t = Y_t = \ln F_t, a = \ln F_{\min}, b = \ln F_{\max}$$

and

$$(\mu(y), \sigma(y)) \in C(y) = \left\{ \left(\mathbf{x}^T \boldsymbol{\pi} - a - \frac{1}{2} \mathbf{x}^T V \mathbf{x}, (\mathbf{x}^T V \mathbf{x})^{0.5} \right) \mid \mathbf{x} \in \mathbb{R}^N \right\} \quad (30)$$

under model specification A.1,

respectively

$$(\mu(y), \sigma(y)) \in C(y) = \left\{ \left((1 - \alpha) \left(\mathbf{x}^T \boldsymbol{\pi} - k(y - \ln \bar{F}) - \frac{1}{2} (1 + \alpha) \mathbf{x}^T V \mathbf{x} \right), (1 - \alpha) (\mathbf{x}^T V \mathbf{x})^{0.5} \right) \mid \mathbf{x} \in \mathbb{R}^N \right\} \quad (31)$$

under model specification A.2.

The shortfall minimizing strategy, respectively $(\mu_0(y), \sigma_0(y))$ results from

$$\max_{\mathbf{x} \in \mathbb{R}^N} \frac{\mathbf{x}^T \boldsymbol{\pi} - a}{\mathbf{x}^T V \mathbf{x}} \quad (32)$$

respectively from

$$\max_{\mathbf{x} \in \mathbb{R}^N} \frac{\mathbf{x}^T \boldsymbol{\pi} - k(y - \ln \bar{F})}{\mathbf{x}^T V \mathbf{x}}. \quad (33)$$

This leads to

Proposition 5

1) Under A.1 the shortfall minimizing investment strategy is given by

$$\mathbf{x}(y) = \frac{2a}{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}} V^{-1} \boldsymbol{\pi}. \quad (34)$$

Moreover

$$\mu_0(y) = a - \frac{2a^2}{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}} \quad (35)$$

$$\sigma_0(y) = \frac{2a}{(\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi})^{0.5}}. \quad (36)$$

2) Under A.2 and $y \geq \ln F_{\min} > \ln \bar{F}$ the shortfall minimizing investment strategy is given by

$$\mathbf{x}(y) = \frac{2k(y - \ln \bar{F})}{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}} V^{-1} \boldsymbol{\pi}. \quad (37)$$

Moreover

$$\mu_0(y) = (1 - \alpha)k(y - \ln \bar{F}) - \frac{2(1 - \alpha^2)k^2(y - \ln \bar{F})^2}{\pi^T V^{-1} \pi} \quad (38)$$

$$\sigma_0(y) = \frac{2(1 - \alpha)k \cdot (y - \ln \bar{F})}{(\pi^T V^{-1} \pi)^{0.5}}. \quad (39)$$

Proof. The proof of part 1 is analogous to the proof of proposition 10.

Part 2 follows directly from proposition 10. ■

Comments

The model specification A.1 which is used as a reference case leads to constant investment strategy and the funding ratio follows a geometric Brownian motion. Under the model specification A.2 the investment policy is highly sensitive to the funding ratio, but does not depend on the participation ratio α .

For $F_{\max} > F_0 > F_{\min} > \bar{F}$ the minimum shortfall probability $P(\tau_{\min} < \tau_{\max})$ can be calculated with the method of Pestien and Sudderth (1985). According to the theorem in subsection 4.2 one obtains

$$\text{Prob}[\tau_{\min} < \tau_{\max}] = \frac{\int_{\ln F_0}^{\ln F_{\max}} \xi(v) dv}{\int_{\ln F_{\min}}^{\ln F_{\max}} \xi(v) dv} \quad (40)$$

with

$$\xi(v) = \exp \left\{ -2 \int_{\ln F_{\min}}^v \frac{\mu_0(z)}{\sigma_0^2(z)} dz \right\}.$$

This leads to

Proposition 6

1) Under A.1 the shortfall probability is

$$\text{Prob}(\tau_{\min} < \tau_{\max}) = \frac{\left(\frac{F_{\max}}{F_{\min}}\right)^{1 - \frac{\pi^T V^{-1} \pi}{2\alpha}} - \left(\frac{F_0}{F_{\min}}\right)^{1 - \frac{\pi^T V^{-1} \pi}{2\alpha}}}{\left(\frac{F_{\max}}{F_{\min}}\right)^{1 - \frac{\pi^T V^{-1} \pi}{2\alpha}} - 1} \quad \text{for } a \neq \frac{\pi^T V^{-1} \pi}{2} \quad (41)$$

$$\text{Prob}(\tau_{\min} < \tau_{\max}) = \frac{\ln F_{\max} - \ln F_0}{\ln F_{\max} - \ln F_{\min}} \quad \text{for } a = \frac{\pi^T V^{-1} \pi}{2}. \quad (42)$$

2) Under A.2 the shortfall probability is

$$\text{Prob}[\tau_{\min} < \tau_{\max}] = \frac{\int_{\ln F_0}^{\ln F_{\max}} (v - \ln \bar{F})^{-\frac{\pi^T V^{-1} \pi}{2(1-\alpha)k}} e^{\frac{1+\alpha}{1-\alpha}v} dv}{\int_{\ln F_{\min}}^{\ln F_{\max}} (v - \ln \bar{F})^{-\frac{\pi^T V^{-1} \pi}{2(1-\alpha)k}} e^{\frac{1+\alpha}{1-\alpha}v} dv}. \quad (43)$$

Proof. Part 1 is proved in appendix B.1.

Part 2 follows from proposition 11. ■

Comment

Part 1 is a standard result on geometric Brownian motions which can be found e.g. in Rogers and Williams (1994), I. 18.

According to part 2 one obtains for $k(1 - \alpha) < \frac{\pi^T V^{-1} \pi}{2}$

$$\lim_{F_{\min} \rightarrow \bar{F}} \text{Prob}(\tau_{\min} < \tau_{\max}) = 0. \quad (44)$$

Hence for F_{\min} close to \bar{F} and $k(1 - \alpha) < \frac{\pi^T V^{-1} \pi}{2}$ the funding ratio attains the level F_{\max} with a probability close to one.

Hence, minimizing the shortfall probability leads to the following main results:

- For a constant return attributed to the accrued retirement benefits (model specification A.1) the investment policy is constant, the funding ratio follows a geometric Brownian motion and the shortfall probability is given by a standard formula.
- If the return attributed to the accrued retirement benefits depends on the funding ratio and on the investment performance (model specification A.2) then the investment policy is very sensitive to changes of the funding ratio but does not depend on the participation rate α . Moreover, for $k < \frac{\pi^T V^{-1} \pi}{2(1-\alpha)}$ the shortfall probability tends to zero if F_{\min} tends to \bar{F} .

5 Role of Net Contributions

In this section model specification A.3 which takes into account non-zero net contributions, is discussed. Hence the dynamics of the funding ratio are given by (8).

In the subsections 5.1, 5.2 we analyze optimal investment policies under expected utility maximization and under shortfall minimization for this extended framework. In particular the impact of net contributions on the optimal investment policy will be discussed.

5.1 Expected Utility Maximization with Net Contributions

As in section 3 the expected utility of the funding ratio F_T at a planning horizon T is maximized. In analogy to proposition 1 we get

Proposition 7

Under model specification A.3 the investment policy \mathbf{x}_t , $0 \leq t \leq T$, maximizing

$$E \left[(1 - R)^{-1} F_T^{1-R} \right]$$

is given by

$$\mathbf{x}_t = \frac{1}{1 + \alpha + (1 - \alpha)(R - 1) e^{[(1-\alpha)k+c](t-T)}} V^{-1} \boldsymbol{\pi}. \quad (45)$$

Proof. See Appendix B.2. ■

Comment

Higher net contributions lead to a more aggressive investment policy.

In analogy to (12) one obtains for $Y_t = \ln F_t$

$$\begin{aligned}
dY_t = & \left\{ -[(1-\alpha)k+c]Y_t + k(1-\alpha)\ln\bar{F} + \frac{(1-\alpha)\boldsymbol{\pi}^T V^{-1}\boldsymbol{\pi}}{1+\alpha+(1-\alpha)(R-1)e^{[(1-\alpha)k+c](t-T)}} \right. \\
& \left. - \frac{1}{2}(1-\alpha^2) \frac{\boldsymbol{\pi}^T V^{-1}\boldsymbol{\pi}}{[1+\alpha+(1-\alpha)(R-1)e^{[(1-\alpha)k+c](t-T)}]^2} \right\} dt \\
& + \frac{(1-\alpha)(\boldsymbol{\pi}^T V^{-1}\boldsymbol{\pi})^{0.5}}{1+\alpha+(1-\alpha)(R-1)e^{[(1-\alpha)k+c](t-T)}} dZ_t^*
\end{aligned} \tag{46}$$

and in analogy to proposition 2 we get

Proposition 8

The solution $Y_t = \ln F_t$ of (46) is Gaussian with

$$\begin{aligned}
E(Y_t) = & e^{-[(1-\alpha)k+c]t} \cdot \left\{ E(Y_0) + \frac{(1-\alpha)k\ln\bar{F}}{(1-\alpha)k+c} \left(e^{[(1-\alpha)k+c]t} - 1 \right) \right. \\
& + \frac{\boldsymbol{\pi}^T V^{-1}\boldsymbol{\pi}}{2[(1-\alpha)k+c](R-1)} e^{[(1-\alpha)k+c]T} \left[2\ln \left(\frac{1+\alpha+(1-\alpha)(R-1)e^{[(1-\alpha)k+c](t-T)}}{1+\alpha+(1-\alpha)(R-1)e^{-[(1-\alpha)k+c]T}} \right) \right. \\
& \left. \left. + \frac{1+\alpha}{1+\alpha+(1-\alpha)(R-1)e^{[(1-\alpha)k+c](t-T)}} - \frac{1+\alpha}{1+\alpha+(1-\alpha)(R-1)e^{-[(1-\alpha)k+c]T}} \right] \right\} \tag{47}
\end{aligned}$$

and

$$\begin{aligned}
Var(Y_t) = & e^{-2[(1-\alpha)k+c]t} \cdot Var(Y_0) + \frac{\boldsymbol{\pi}^T V^{-1}\boldsymbol{\pi}}{[(1-\alpha)k+c](R-1)^2} e^{2[(1-\alpha)k+c](T-t)} \cdot \\
& \left[\frac{1+\alpha}{1+\alpha+(1-\alpha)(R-1)e^{[(1-\alpha)k+c](t-T)}} - \frac{1+\alpha}{1+\alpha+(1-\alpha)(R-1)e^{-[(1-\alpha)k+c]T}} \right. \\
& \left. + \ln \left(\frac{1+\alpha+(1-\alpha)(R-1)e^{[(1-\alpha)k+c](t-T)}}{1+\alpha+(1-\alpha)(R-1)e^{-[(1-\alpha)k+c]T}} \right) \right].
\end{aligned} \tag{48}$$

Proof. See Appendix B.3. ■

The properties of the funding ratio at the investment horizon T are given by

Proposition 9

Assume $(1-\alpha)k+c > 0$. Then

1) $E(Y_T)$ and $Var(Y_T)$ are strictly decreasing in R .

2)

$$\begin{aligned}
\lim_{T \rightarrow \infty} E(Y_T) = & \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln\bar{F} + \frac{\boldsymbol{\pi}^T V^{-1}\boldsymbol{\pi}}{[(1-\alpha)k+c](R-1)} \ln \left(\frac{1+\alpha+(1-\alpha)(R-1)}{1+\alpha} \right) \\
& - \frac{1-\alpha}{2[(1-\alpha)k+c]} \cdot \frac{\boldsymbol{\pi}^T V^{-1}\boldsymbol{\pi}}{1+\alpha+(1-\alpha)(R-1)}
\end{aligned} \tag{49}$$

$$\begin{aligned}
\lim_{T \rightarrow \infty} Var(Y_T) = & \frac{\boldsymbol{\pi}^T V^{-1}\boldsymbol{\pi}}{[(1-\alpha)k+c](R-1)^2} \ln \left(\frac{1+\alpha+(1-\alpha)(R-1)}{1+\alpha} \right) \\
& - \frac{1-\alpha}{[(1-\alpha)k+c](R-1)} \cdot \frac{\boldsymbol{\pi}^T V^{-1}\boldsymbol{\pi}}{1+\alpha+(1-\alpha)(R-1)}
\end{aligned} \tag{50}$$

3)

$$\lim_{T \rightarrow \infty} E(Y_T) > \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F}. \quad (51)$$

4) For $\alpha = 0$

$$\begin{aligned} \lim_{T \rightarrow \infty} E(Y_T) &= \frac{k}{k+c} \ln \bar{F} + \frac{\pi^T V^{-1} \pi}{k+c} \left(\frac{\ln R}{R-1} - \frac{1}{2R} \right) \\ \lim_{T \rightarrow \infty} Var(Y_T) &= \frac{\pi^T V^{-1} \pi}{(k+c)(R-1)} \left(\frac{\ln R}{R-1} - \frac{1}{R} \right). \end{aligned} \quad (52)$$

5) For $\alpha \rightarrow 1, c > 0$

$$\lim_{T \rightarrow \infty} E(Y_T) \rightarrow 0, \lim_{T \rightarrow \infty} Var(Y_T) \rightarrow 0.$$

For $\alpha \rightarrow 1, c = 0$

$$\lim_{T \rightarrow \infty} E(Y_T) \rightarrow \ln \bar{F} + \frac{\pi^T V^{-1} \pi}{4k}, \lim_{T \rightarrow \infty} Var(Y_T) \rightarrow 0. \quad (53)$$

Proof. See Appendix B.4. ■

Comment

From (49), (52) it can be seen that for $\bar{F} < 1$ the effect of net contributions on $\lim_{T \rightarrow \infty} E(Y_T)$ is ambiguous. Positive net contributions draw $\lim_{T \rightarrow \infty} E(Y_T)$ towards zero. The term $\frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F}$ reflects the fact that for a given investment policy positive net contributions draw the funding ratio closer to one.

According to (50) $\lim_{T \rightarrow \infty} Var(Y_T)$ decreases with higher net contributions.

Hence under expected utility maximization net contributions have the following main effects:

- Positive net contributions ($c > 0$) lead to a more aggressive investment policy.
- Positive net contributions ($c > 0$) pull $\lim_{T \rightarrow \infty} E(\ln F_T)$ and $\lim_{T \rightarrow \infty} Var(\ln F_T)$ towards zero.

5.2 Minimization of the Shortfall Probability with Net Contributions

Obviously equation (30), respectively (31) has to be modified to

$$\begin{aligned} (\mu(y), \sigma(y)) \in C(y) = & \left\{ \left((1-\alpha) \left(\mathbf{x}^T \boldsymbol{\pi} - \left(k + \frac{c}{1-\alpha} \right) y + k \ln \bar{F} - \frac{1}{2} (1+\alpha) \mathbf{x}^T V \mathbf{x} \right), \right. \right. \\ & \left. \left. (1-\alpha) (\mathbf{x}^T V \mathbf{x})^{0.5} \right) \mid \mathbf{x} \in \mathbb{R}^N \right\}. \end{aligned} \quad (54)$$

The shortfall minimizing strategy, respectively $(\mu_0(y), \sigma_0(y))$ result from

$$\max_{\mathbf{x} \in \mathbb{R}^N} \frac{\pi^T \mathbf{x} - \left(k + \frac{c}{1-\alpha} \right) y + k \ln \bar{F}}{\mathbf{x}^T V \mathbf{x}}. \quad (55)$$

This leads to

Proposition 10 Under A.3 and $y \geq \ln F_{\min} > \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F}$ the shortfall minimizing investment strategy is given by

$$\mathbf{x}(y) = \frac{2 \left(k + \frac{c}{1-\alpha} \right) \left(y - \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F} \right)}{\pi^T V^{-1} \pi} V^{-1} \boldsymbol{\pi}. \quad (56)$$

Moreover

$$\mu_0(y) = [(1-\alpha)k+c] \left(y - \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F} \right) \quad (57)$$

$$-2(1-\alpha^2) \left(k + \frac{c}{1-\alpha} \right)^2 \left(y - \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F} \right)^2 (\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi})^{-1}$$

$$\sigma_0(y) = 2[(1-\alpha)k+c] \left(y - \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F} \right) (\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi})^{-0.5}. \quad (58)$$

Proof. See Appendix B.5. ■

Comment

Inserting $\ln F_t = \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F}$, $\mathbf{x}_t = \mathbf{0}$ into (8) leads to $dF_t = 0$. Therefore formula (56) for the shortfall minimizing investment policy is quite natural. For a funding ratio larger (smaller) than one, i.e. $y > 0$ ($y < 0$), higher net contributions lead to a more (less) aggressive investment strategy.

In analogy to proposition 6 one obtains

Proposition 11 *Under A.3 the shortfall probability is*

$$\text{Prob}(\tau_{\min} < \tau_{\max}) = \frac{\int_{\ln F_0}^{\ln F_{\max}} \left(v - \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F} \right)^{-\frac{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}}{2[(1-\alpha)k+c]}} \cdot e^{\frac{1+\alpha}{1-\alpha}v} dv}{\int_{\ln F_{\min}}^{\ln F_{\max}} \left(v - \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F} \right)^{-\frac{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}}{2[(1-\alpha)k+c]}} \cdot e^{\frac{1+\alpha}{1-\alpha}v} dv}, \quad (59)$$

where $\ln F_{\max} > \ln F_0 > \ln F_{\min} > \frac{(1-\alpha)k}{(1-\alpha)k+c} \cdot \ln \bar{F}$.

Proof. See Appendix B.6. ■

For $(1-\alpha)k+c < \frac{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}}{2}$ formula (44) becomes

$$\lim_{\ln F_{\min} \rightarrow \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F}} \text{Prob}(\tau_{\min} < \tau_{\max}) = 0. \quad (60)$$

The results on the minimization of the shortfall probability may be interpreted as follows:

- Positive net contributions ($c > 0$) pull $\ln F_t$ towards zero.
- Therefore positive net contributions ($c > 0$) lead to a less aggressive shortfall minimizing strategy for $0 > \ln F_t > \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F}$ and to a more aggressive strategy for $\ln F_t > 0$.
- For $\ln F_t < \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F}$ even a riskless strategy would increase the funding ratio in the case of positive net contributions ($c > 0$).

6 Conclusion

In this paper we analyzed optimal investment strategies for a defined contribution pension plan. Crucial for the model were the assumptions with respect to the return attributed by the fund on the accrued retirement benefits of employees. For this attribution rate we used three different model specifications (A.1, A.2, A.3). The model specification A.1, where the attribution rate is a constant, was used as a

reference case. Under model specification A.2 the attribution rate depends on the funding ratio and partly on the investment performance of the fund. Model specification A.3 admits non-zero net contributions to the fund.

Under these model specifications we dealt first with expected utility maximization of the funding ratio at the investment horizon. Model specification A.1 leads to the Merton portfolio. For model specification A.2 the investment strategy is more aggressive than the Merton portfolio, provided the relative risk aversion is at least two. The participation of employees in the investment performance of the fund has an ambiguous effect on the optimal investment policy. This can be explained by the fact that such a participation leads to positive correlation between the growth rates of assets and liabilities. Under model specification A.3 positive (negative) net contributions lead to a more (less) aggressive investment strategy. The funding ratio is lognormally distributed and asymptotically well behaved for all model specifications. For the minimization of the shortfall probability a technique developed by Pestien and Sudderth (1985) was used. Model specification A.1 leads to constant portfolio weights and the funding ratio follows a geometric Brownian motion. For model specifications A.2 and A.3 the optimal portfolios are highly sensitive to the funding ratio. Under A.2 the participation of employees in the investment performance has no influence on the optimal investment strategy. For all model specifications shortfall probabilities were calculated.

Appendix B.1

Proof of proposition 6

From (35), (36) and (40) one concludes

$$\begin{aligned}\xi(v) &= \exp \left\{ -2 \int_{\ln F_{\min}}^v \left(\frac{\pi^T V^{-1} \pi}{4a} - \frac{1}{2} \right) dz \right\} \\ &= \exp \left\{ \left(1 - \frac{\pi^T V^{-1} \pi}{2a} \right) (v - \ln F_{\min}) \right\}\end{aligned}$$

and

$$\begin{aligned}\text{Prob}(\tau_{\min} < \tau_{\max}) &= \frac{\int_{\ln F_0}^{\ln F_{\max}} e^{\left(1 - \frac{\pi^T V^{-1} \pi}{2a}\right)(v - \ln F_{\min})} dv}{\int_{\ln F_{\min}}^{\ln F_{\max}} e^{\left(1 - \frac{\pi^T V^{-1} \pi}{2a}\right)(v - \ln F_{\min})} dv} \\ &= \frac{e^{\left(1 - \frac{\pi^T V^{-1} \pi}{2a}\right)(\ln F_{\max} - \ln F_{\min})} - e^{\left(1 - \frac{\pi^T V^{-1} \pi}{2a}\right)(\ln F_0 - \ln F_{\min})}}{e^{\left(1 - \frac{\pi^T V^{-1} \pi}{2a}\right)(\ln F_{\max} - \ln F_{\min})} - 1} \\ &= \frac{\left(\frac{F_{\max}}{F_{\min}}\right)^{1 - \frac{\pi^T V^{-1} \pi}{2a}} - \left(\frac{F_0}{F_{\min}}\right)^{1 - \frac{\pi^T V^{-1} \pi}{2a}}}{\left(\frac{F_{\max}}{F_{\min}}\right)^{1 - \frac{\pi^T V^{-1} \pi}{2a}} - 1}.\end{aligned}$$

Appendix B.2

Proof of proposition 7

The dynamics of the funding ratio F_t are given by

$$dF_t = F_t (1 - \alpha) \left[\mathbf{x}_t^T \boldsymbol{\pi} - \left(k + \frac{c}{1 - \alpha} \right) \ln F_t + k \ln \bar{F} - \alpha \mathbf{x}_t^T V \mathbf{x}_t \right] dt + (1 - \alpha) F_t \mathbf{x}_t^T \sigma d\mathbf{Z}_t.$$

This leads to the HJB-equation

$$0 = J_t + \max_{\mathbf{x}} \left\{ J_F \cdot F_t (1 - \alpha) \left[\mathbf{x}_t^T \boldsymbol{\pi} - \left(k + \frac{c}{1 - \alpha} \right) \ln F_t + k \ln \bar{F} - \alpha \mathbf{x}_t^T V \mathbf{x}_t \right] + \frac{1}{2} J_{FF} F_t^2 (1 - \alpha)^2 \mathbf{x}_t^T V \mathbf{x}_t \right\}$$

with

$$J(T, F) = (1 - R)^{-1} F^{1-R}, \quad R > 1.$$

The optimal investment strategy results from

$$(1 - \alpha) J_F \cdot F_t [\boldsymbol{\pi} - 2\alpha V \mathbf{x}_t] + J_{FF} F_t^2 (1 - \alpha)^2 V \mathbf{x}_t = \mathbf{0}$$

$$J_F \cdot \boldsymbol{\pi} - [2\alpha J_F - (1 - \alpha) J_{FF} F_t] V \mathbf{x}_t = \mathbf{0}$$

$$\mathbf{x}_t = \frac{J_F}{2\alpha J_F - (1 - \alpha) J_{FF} F_t} V^{-1} \boldsymbol{\pi}.$$

Inserting into the HJB-equation leads to

$$0 = J_t + \frac{1}{2} J_F \cdot F_t (1 - \alpha) \frac{J_F}{2\alpha J_F - (1 - \alpha) J_{FF} \cdot F_t} \boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi} - J_F \cdot F_t \left([(1 - \alpha) k + c] \ln F_t - (1 - \alpha) k \ln \bar{F} \right).$$

For the solution one tries

$$J(t, F) = -e^{g(t)} F^{h(t)}.$$

Using

$$J_F \cdot F = h(t) J$$

$$J_{FF} \cdot F^2 = h(t) [h(t) - 1] \cdot J$$

$$J_t = [g'(t) + h'(t) \ln F] \cdot J$$

one obtains

$$0 = g'(t) + h'(t) \cdot \ln F + \frac{1}{2} h(t) (1 - \alpha) \frac{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}}{2\alpha + (1 - \alpha) [1 - h(t)]} - h(t) [(1 - \alpha) k + c] \ln F + (1 - \alpha) h(t) k \ln \bar{F}.$$

From

$$h'(t) = [(1 - \alpha)k + c] h(t)$$

$$h(T) = 1 - R$$

one concludes

$$h(t) = (1 - R) e^{[(1-\alpha)k+c](t-T)}, \quad 0 \leq t \leq T.$$

There is no need to calculate $g(t)$.

One obtains

$$\begin{aligned} \mathbf{x}_t &= \frac{1}{2\alpha + (1 - \alpha)[1 - h(t)]} V^{-1} \boldsymbol{\pi} \\ &= \frac{1}{2\alpha + (1 - \alpha)[1 + (R - 1) e^{[(1-\alpha)k+c](t-T)}]} V^{-1} \boldsymbol{\pi} \end{aligned}$$

or

$$\mathbf{x}_t = \frac{1}{1 + \alpha + (1 - \alpha)(R - 1) e^{[(1-\alpha)k+c](t-T)}} V^{-1} \boldsymbol{\pi}.$$

Appendix B.3

Proof of proposition 8

Equation (46) can be written in the form

$$\begin{aligned} dY_t &= \left[-AY_t + B + \frac{C}{D + E \cdot e^{A(t-T)}} - \frac{G}{[D + E \cdot e^{A(t-T)}]^2} \right] dt \\ &\quad + \frac{H}{D + E \cdot e^{A(t-T)}} dZ_t^*, \end{aligned}$$

where

$$\begin{aligned} A &= (1 - \alpha)k + c, \quad B = k(1 - \alpha) \ln \bar{F}, \quad C = (1 - \alpha) \boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}, \\ D &= 1 + \alpha, \quad E = (1 - \alpha)(R - 1), \quad G = \frac{1}{2} (1 - \alpha^2) \boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}, \\ H &= (1 - \alpha) (\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi})^{0.5}. \end{aligned}$$

Using the notation

$$\begin{aligned} a(t) &= B + \frac{C}{D + E \cdot e^{A(t-T)}} - \frac{G}{[D + E \cdot e^{A(t-T)}]^2} \\ \sigma(t) &= \frac{H}{D + E \cdot e^{A(t-T)}} \end{aligned}$$

the solution is given by

$$Y_t = \Phi(t) \left[Y_0 + \int_0^t [\Phi(s)]^{-1} a(s) ds + \int_0^t [\Phi(s)]^{-1} \sigma(s) dZ_s^* \right]$$

with

$$\Phi(t) = e^{-At}.$$

Hence Y_t is a Gaussian stochastic process with

$$\begin{aligned}
E(Y_t) &= e^{-At} \left[E(Y_0) + \int_0^t e^{As} \cdot a(s) ds \right] \\
E(Y_t) &= e^{-At} \left[E(Y_0) + B \int_0^t e^{As} ds + C \int_0^t \frac{e^{As}}{D + E \cdot e^{A(s-T)}} ds \right. \\
&\quad \left. - G \int_0^t \frac{e^{As}}{[D + E \cdot e^{A(s-T)}]^2} ds \right] \\
E(Y_t) &= e^{-At} \left[E(Y_0) + \frac{B}{A} (e^{At} - 1) + \frac{Ce^{AT}}{AE} \cdot \ln \left(D + E \cdot e^{A(s-T)} \right) \Big|_0^t \right. \\
&\quad \left. + \frac{Ge^{AT}}{AE} \frac{1}{D + E \cdot e^{A(s-T)}} \Big|_0^t \right] \\
E(Y_t) &= e^{-At} \left[E(Y_0) + \frac{B}{A} (e^{At} - 1) + \frac{Ce^{AT}}{AE} \cdot \ln \left(\frac{D + E \cdot e^{A(t-T)}}{D + E \cdot e^{-AT}} \right) \right. \\
&\quad \left. + \frac{Ge^{AT}}{AE} \left(\frac{1}{D + E \cdot e^{A(t-T)}} - \frac{1}{D + E \cdot e^{-AT}} \right) \right].
\end{aligned}$$

Inserting A, B, C, D, E, G proves (47).

Moreover

$$\begin{aligned}
Var(Y_t) &= e^{-2At} \cdot Var(Y_0) + e^{-2At} H^2 \int_0^t \frac{e^{2As}}{(D + E \cdot e^{A(s-T)})^2} ds \\
Var(Y_t) &= e^{-2At} \cdot Var(Y_0) + \frac{e^{-2At} \cdot H^2}{A} \cdot \int_1^{e^{At}} \frac{x}{(D + E \cdot e^{-AT} \cdot x)^2} dx
\end{aligned}$$

Using formula 6, p. 35 in Bronstein and Semendjajew (1991) one obtains

$$\begin{aligned}
Var(Y_t) &= e^{-2At} \cdot Var(Y_0) + \frac{H^2 e^{2A(T-t)}}{AE^2} \cdot \left(\frac{D}{D + E \cdot e^{-AT} \cdot x} + \ln(D + E \cdot e^{-AT} \cdot x) \right) \Big|_1^{e^{At}} \\
&= e^{-2At} \cdot Var(Y_0) + \frac{H^2 e^{2A(T-t)}}{AE^2} \\
&\quad \cdot \left(\frac{D}{D + E \cdot e^{A(t-T)}} - \frac{D}{D + E \cdot e^{-AT}} + \ln \left(\frac{D + E \cdot e^{A(t-T)}}{D + E \cdot e^{-AT}} \right) \right).
\end{aligned}$$

Inserting A, D, E, H leads to (48).

Appendix B.4

Proof of proposition 9

a) From (46) it is obvious that $Var(Y_T)$ is strictly decreasing in R . Moreover, as argued in the first comment on proposition 3 expected utility maximization is equivalent to mean variance optimization with respect to Y_T . This proves part 1.

b) Part 2 follows directly from (47) and (48).

c) In order to prove part 3 one has to show

$$\ln\left(\frac{1+\alpha+(1-\alpha)(R-1)}{1+\alpha}\right) - \frac{(1-\alpha)(R-1)}{2(1+\alpha)} \cdot \frac{1+\alpha}{1+\alpha+(1-\alpha)(R-1)} > 0$$

respectively

$$\ln\left(1 + \frac{(1-\alpha)(R-1)}{1+\alpha}\right) - \frac{1}{2} \cdot \frac{(1-\alpha)(R-1)}{1+\alpha} \cdot \frac{1}{1 + \frac{(1-\alpha)(R-1)}{1+\alpha}} > 0.$$

Hence we have to prove

$$f(x) = \ln(1+x) - \frac{1}{2} \frac{x}{1+x} > 0 \text{ for } x > 0.$$

However this inequality follows immediately from

$$f(0) = 0, f'(x) = \frac{1}{1+x} - \frac{1}{2} \frac{1}{(1+x)^2} > 0 \text{ for } x > 0.$$

d) (52) follows directly from (49) and (50).

e) Proof of (53)

e.1) For $\alpha \rightarrow 1, c > 0$ one obtains

$$\begin{aligned} \lim_{T \rightarrow \infty} E(Y_T) &\rightarrow 0 \\ \lim_{T \rightarrow \infty} Var(Y_T) &\rightarrow 0. \end{aligned}$$

e.2) For $\alpha \rightarrow 1, c = 0$ one obtains

$$\begin{aligned} \lim_{T \rightarrow \infty} E(Y_T) &\rightarrow \ln \bar{F} + \frac{\pi^T V^{-1} \pi}{k(R-1)} \cdot \lim_{\alpha \rightarrow 1} \left[\frac{\ln(1+\alpha+(1-\alpha)(R-1)) - \ln(1+\alpha)}{1-\alpha} \right] \\ &\quad - \frac{1}{2k} \frac{\pi^T V^{-1} \pi}{2} \\ &= \ln \bar{F} + \frac{\pi^T V^{-1} \pi}{k(R-1)} \frac{\frac{2-R}{2} - \frac{1}{2}}{-1} - \frac{\pi^T V^{-1} \pi}{4k} \\ &= \ln \bar{F} + \frac{\pi^T V^{-1} \pi}{4k} \end{aligned}$$

$$\begin{aligned} \lim_{T \rightarrow \infty} Var(Y_T) &\rightarrow \frac{\pi^T V^{-1} \pi}{k(R-1)^2} \cdot \lim_{\alpha \rightarrow 1} \left[\frac{\ln(1+\alpha+(1-\alpha)(R-1)) - \ln(1+\alpha)}{1-\alpha} \right] \\ &\quad - \frac{1}{k(R-1)} \frac{\pi^T V^{-1} \pi}{2} \\ &= \frac{\pi^T V^{-1} \pi}{k(R-1)^2} \frac{\frac{2-R}{2} - \frac{1}{2}}{-1} - \frac{1}{k(R-1)} \frac{\pi^T V^{-1} \pi}{2} \\ &= 0. \end{aligned}$$

Appendix B.5

Proof of proposition 10

For $y > \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F}$ one can easily show that the optimization problem (55) has an optimal solution of the form

$$\mathbf{x} = \lambda V^{-1} \boldsymbol{\pi}.$$

Inserting into (55) leads to

$$\max_{\lambda > 0} \frac{\lambda \boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi} - \left(k + \frac{c}{1-\alpha}\right) y + k \ln \bar{F}}{\lambda^2 \boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}}$$

or

$$\max_{\lambda > 0} \left(\frac{1}{\lambda} - \frac{1}{\lambda^2} \frac{\left(k + \frac{c}{1-\alpha}\right) y - k \ln \bar{F}}{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}} \right).$$

The maximum is attained at

$$\lambda = 2 \frac{\left(k + \frac{c}{1-\alpha}\right) y - k \ln \bar{F}}{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}}.$$

Inserting the optimal portfolio

$$\mathbf{x} = 2 \frac{\left(k + \frac{c}{1-\alpha}\right) y - k \ln \bar{F}}{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}} V^{-1} \boldsymbol{\pi}.$$

into (54) leads to (57) and (58).

Appendix B.6

Proof of proposition 11

From (57) and (58) one obtains

$$\frac{\mu_0(y)}{\sigma_0^2(y)} = \frac{1}{4} \frac{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}}{[(1-\alpha)k + c] \left(y - \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F}\right)} - \frac{1}{2} \cdot \frac{1+\alpha}{1-\alpha}.$$

According to (40) this leads to

$$\begin{aligned} \xi(v) &= \exp \left\{ -2 \int_{\ln F_{\min}}^v \left(\frac{1}{4} \cdot \frac{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}}{[(1-\alpha)k + c] \left(z - \frac{(1-\alpha)k}{(1-\alpha)k+c} \cdot \ln \bar{F}\right)} - \frac{1}{2} \frac{1+\alpha}{1-\alpha} \right) dz \right\} \\ &= \exp \left\{ \left[-\frac{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}}{2[(1-\alpha)k + c]} \cdot \ln \left(z - \frac{(1-\alpha)k}{(1-\alpha)k+c} \ln \bar{F} \right) + \frac{1+\alpha}{1-\alpha} z \right]_{\ln F_{\min}}^v \right\} \\ &= \exp \left\{ -\frac{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}}{2[(1-\alpha)k + c]} \cdot \ln \left(\frac{v - \frac{(1-\alpha)k}{(1-\alpha)k+c} \cdot \ln \bar{F}}{\ln F_{\min} - \frac{(1-\alpha)k}{(1-\alpha)k+c} \cdot \ln \bar{F}} \right) + \frac{1+\alpha}{1-\alpha} (v - \ln F_{\min}) \right\} \\ &= \left(\frac{v - \frac{(1-\alpha)k}{(1-\alpha)k+c} \cdot \ln \bar{F}}{\ln F_{\min} - \frac{(1-\alpha)k}{(1-\alpha)k+c} \cdot \ln \bar{F}} \right)^{-\frac{\boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}}{2[(1-\alpha)k + c]}} \cdot e^{\frac{1+\alpha}{1-\alpha} (v - \ln F_{\min})} \end{aligned}$$

Inserting this formula into (40) leads to (59).

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